

# The $C^*$ -algebras of connected real two-step nilpotent Lie groups

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## Abstract

Using the operator valued Fourier transform, the  $C^*$ -algebras of connected real two-step nilpotent Lie groups are characterized as algebras of operator fields defined over their spectra. In particular, it is shown by explicit computations, that the Fourier transform of such  $C^*$ -algebras fulfills the norm controlled dual limit property.

## 1 Introduction

In this article, the structure of the  $C^*$ -algebras of two-step nilpotent Lie groups will be analyzed. In order to be able to understand these  $C^*$ -algebras, the Fourier transform is an important tool. The Fourier transform  $\mathcal{F}(a) = \hat{a}$  of an element  $a$  of a  $C^*$ -algebra  $A$  is defined in the following way: One chooses for every  $\gamma$  in  $\widehat{A}$ , the spectrum of  $A$ , a representation  $(\pi_\gamma, \mathcal{H}_\gamma)$  in the equivalence class of  $\gamma$  and defines

$$\mathcal{F}(a)(\gamma) := \pi_\gamma(a) \in \mathcal{H}_\gamma \quad \forall \gamma \in \widehat{A}.$$

Then  $\mathcal{F}(a)$  is contained in the algebra of all bounded operator fields over  $\widehat{A}$

$$l^\infty(\widehat{A}) = \left\{ \phi = (\phi(\pi_\gamma) \in \mathcal{B}(\mathcal{H}_\gamma))_{\gamma \in \widehat{A}} \mid \|\phi\|_\infty := \sup_{\gamma \in \widehat{A}} \|\phi(\pi_\gamma)\|_{op} < \infty \right\}$$

and the mapping

$$\mathcal{F} : A \rightarrow l^\infty(\widehat{A}), \quad a \mapsto \hat{a}$$

is an isometric  $*$ -homomorphism.

The structure of the  $C^*$ -algebras is already known for certain classes of Lie groups: The  $C^*$ -algebras of the Heisenberg and the thread-like Lie groups have been characterized in [7] and the  $C^*$ -algebras of the  $ax + b$ -like groups in [6]. Furthermore, the  $C^*$ -algebras of the 5-dimensional nilpotent Lie groups have been determined in [11] and H.Regeiba analyzed the  $C^*$ -algebras of all 6-dimensional nilpotent Lie groups in his doctoral thesis (see [10]).

The methods in this paper will partly be similar, but more complex, to the one used for the characterization of the  $C^*$ -algebra of the Heisenberg Lie group (see [7]), which is also two-step nilpotent and thus serves as an example.

It will be shown that the  $C^*$ -algebras of two-step nilpotent Lie groups  $G$  are characterized by the following conditions. The same conditions hold true for all 5- and 6-dimensional nilpotent Lie groups (see [11]), for the Heisenberg Lie groups and the thread-like Lie groups (see [7]).

## Conditions 1.1.

### 1. Stratification of the spectrum:

- (a) A finite increasing family  $S_0 \subset S_1 \subset \dots \subset S_r = \widehat{C^*(G)} \cong \widehat{G}$  of closed subsets of the spectrum  $\widehat{C^*(G)} \cong \widehat{G}$  of  $C^*(G)$  or respectively  $G$  will be constructed in such a way that for  $i \in \{1, \dots, r\}$  the subsets  $\Gamma_0 = S_0$  and  $\Gamma_i := S_i \setminus S_{i-1}$  are Hausdorff in their relative topologies and such that  $S_0$  consists of all the characters of  $C^*(G)$  or  $G$ , respectively.
- (b) For every  $i \in \{0, \dots, r\}$  a Hilbert space  $\mathcal{H}_i$  and for every  $\gamma \in \Gamma_i$  a concrete realization  $(\pi_\gamma, \mathcal{H}_i)$  of  $\gamma$  on the Hilbert space  $\mathcal{H}_i$  will be defined.

### 2. CCR $C^*$ -algebra:

It will be shown that  $C^*(G)$  is a separable CCR (or liminal)  $C^*$ -algebra, i.e. a separable  $C^*$ -algebra such that the image of every irreducible representation  $(\pi, \mathcal{H})$  of  $C^*(G)$  is contained in the algebra of compact operators  $\mathcal{K}(\mathcal{H})$  (which implies that the image equals  $\mathcal{K}(\mathcal{H})$ ).

### 3. Changing of layers:

Let  $a \in C^*(G)$ .

- (a) It will be proved that the mappings  $\gamma \mapsto \mathcal{F}(a)(\gamma)$  are norm continuous on the different sets  $\Gamma_i$ .
- (b) For any  $i \in \{0, \dots, r\}$  and for any converging sequence contained in  $\Gamma_i$  with limit set outside  $\Gamma_i$  (thus in  $S_{i-1}$ ), there will be constructed a properly converging subsequence  $\bar{\gamma} = (\gamma_k)_{k \in \mathbb{N}}$  (i.e. the subsequences of  $\bar{\gamma}$  have all the same limit set - see Definition 2.2), as well as a constant  $C > 0$  and for every  $k \in \mathbb{N}$  an involutive linear mapping  $\tilde{\nu}_k = \tilde{\nu}_{\bar{\gamma}, k} : CB(S_{i-1}) \rightarrow \mathcal{B}(\mathcal{H}_i)$ , which is bounded by  $C \|\cdot\|_{S_{i-1}}$ , such that

$$\lim_{k \rightarrow \infty} \|\mathcal{F}(a)(\gamma_k) - \tilde{\nu}_k(\mathcal{F}(a)|_{S_{i-1}})\|_{op} = 0.$$

Here  $CB(S_{i-1})$  is the  $*$ -algebra of all the uniformly bounded fields of operators  $(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_j))_{\gamma \in \Gamma_j, j=0, \dots, i-1}$ , which are operator norm continuous on the subsets  $\Gamma_j$  for every  $j \in \{0, \dots, i-1\}$ , provided with the infinity-norm

$$\|\varphi\|_{S_{i-1}} := \sup_{\gamma \in S_{i-1}} \|\varphi(\gamma)\|_{op}.$$

These properties characterize the structure of  $C^*(G)$  (see [11], Theorem 3.5). A  $C^*$ -algebra fulfilling these conditions is called a  $C^*$ -algebra with "norm controlled dual limits".

The main work of this article consists in the proof of Property 3(b) and in particular in the construction of the mappings  $(\tilde{\nu}_k)_k$ .

## 2 Preliminaries

### 2.1 Two-step nilpotent Lie groups

Let  $\mathfrak{g}$  be a real Lie algebra which is nilpotent of step two. This means that

$$[\mathfrak{g}, \mathfrak{g}] := \text{span}\{[X, Y] \mid X, Y \in \mathfrak{g}\}$$

is contained in the center of  $\mathfrak{g}$ .

Fix a scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  and take on  $\mathfrak{g}$  the Campbell-Baker-Hausdorff multiplication

$$u \cdot v = u + v + \frac{1}{2}[u, v] \quad \forall u, v \in \mathfrak{g}.$$

This gives the simply connected connected Lie group  $G = (\mathfrak{g}, \cdot)$  with Lie algebra  $\mathfrak{g}$ . The exponential mapping  $\exp : \mathfrak{g} \rightarrow G = (\mathfrak{g}, \cdot)$  is in this case the identity mapping.

The Haar measure of this group is a Lebesgue measure which is denoted by  $dx$ . Then, the  $C^*$ -algebra of  $G$  is defined as the completion of the convolution algebra  $L^1(G, dx) = L^1(G)$  with respect to the  $C^*$ -norm of  $L^1(G, dx)$ , i.e.

$$C^*(G) := \overline{L^1(G, dx)}^{\|\cdot\|_{C^*(G)}} \quad \text{with} \quad \|f\|_{C^*(G)} := \sup_{\pi \in \widehat{G}} \|\pi(f)\|_{op}$$

and a well-known result, that can be found in [3], states that the spectrum of  $C^*(G)$  coincides with the spectrum of  $G$ :

$$\widehat{C^*(G)} = \widehat{G}.$$

Now, for a linear functional  $\ell$  of  $\mathfrak{g}$ , consider the skew-bilinear form

$$B_\ell(X, Y) := \langle \ell, [X, Y] \rangle$$

on  $\mathfrak{g}$ . Moreover, let

$$\mathfrak{g}(\ell) := \{X \in \mathfrak{g} \mid \langle \ell, [X, \mathfrak{g}] \rangle = \{0\}\}$$

be the radical of  $B_\ell$  and the stabilizer of the linear functional  $\ell$ . Then, as  $\mathfrak{g}$  is two-step nilpotent,  $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}(\ell)$  and thus  $\mathfrak{g}(\ell)$  is an ideal of  $\mathfrak{g}$ .

**Definition 2.1.**

A subalgebra  $\mathfrak{p}$  of  $\mathfrak{g}$ , that is *subordinated* to  $\ell$  (i.e. that fulfills  $\langle \ell, [\mathfrak{p}, \mathfrak{p}] \rangle = \{0\}$ ) and that has the dimension

$$\dim(\mathfrak{p}) = \frac{1}{2}(\dim(\mathfrak{g}) + \dim(\mathfrak{g}(\ell))),$$

which means that  $\mathfrak{p}$  is maximal isotropic for  $B_\ell$ , is called a *polarization* in  $\ell$ .

Again since  $\mathfrak{g}$  is nilpotent of step two, every maximal isotropic subspace  $\mathfrak{p}$  of  $\mathfrak{g}$  for  $B_\ell$  containing  $[\mathfrak{g}, \mathfrak{g}]$  is a polarization at  $\ell$ .

Now, if  $\mathfrak{p} \subset \mathfrak{g}$  is any subalgebra of  $\mathfrak{g}$  which is subordinated to  $\ell$ , the linear functional  $\ell$  defines a unitary character  $\chi_\ell$  of  $P := \exp(\mathfrak{p})$ :

$$\chi_\ell(x) := e^{-2\pi i \langle \ell, \log(x) \rangle} = e^{-2\pi i \langle \ell, x \rangle} \quad \forall x \in P.$$

## 2.2 Induced representations

The induced representation  $\sigma_{\ell, \mathfrak{p}} = \text{ind}_P^G \chi_\ell$  for a polarization  $\mathfrak{p}$  in  $\ell$  and  $P := \exp(\mathfrak{p})$  can be described in the following way:

Since  $\mathfrak{p}$  contains  $[\mathfrak{g}, \mathfrak{g}]$  and even the center  $\mathfrak{z}$  of  $\mathfrak{g}$ , one can write  $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{p}$  and  $\mathfrak{p} = \mathfrak{t} \oplus \mathfrak{z}$  for two subspaces  $\mathfrak{t}$  and  $\mathfrak{s}$  of  $\mathfrak{g}$ . The quotient space  $G/P$  is then homeomorphic to  $\mathfrak{s}$  and the Lebesgue measure  $ds$  on  $\mathfrak{s}$  defines an invariant Borel measure  $d\dot{g}$  on  $G/P$ . The group  $G$  acts by the left translation  $\sigma_{\ell, \mathfrak{p}}$  on the Hilbert space

$$L^2(G/P, \chi_\ell) := \left\{ \xi : G \rightarrow \mathbb{C} \mid \xi \text{ measurable, } \xi(gp) = \overline{\chi_\ell(p)} \xi(g) \quad \forall g \in G \quad \forall p \in P, \right. \\ \left. \|\xi\|_2^2 := \int_{G/P} |\xi(g)|^2 d\dot{g} < \infty \right\}.$$

Now, if one uses the coordinates  $G = \mathfrak{s} \cdot \mathfrak{p}$ , one can identify the Hilbert spaces  $L^2(G/P, \chi_\ell)$  and  $L^2(\mathfrak{s}) = L^2(\mathfrak{s}, ds)$ :

Let  $U_\ell : L^2(\mathfrak{s}, ds) \rightarrow L^2(G/P, \chi_\ell)$  be defined by

$$U_\ell(\varphi)(S \cdot Y) := \chi_\ell(-Y)\varphi(S) \quad \forall Y \in \mathfrak{p} \quad \forall S \in \mathfrak{s} \quad \forall \varphi \in L^2(\mathfrak{s}).$$

Then,  $U_\ell$  is a unitary operator and one can transform the representation  $\sigma_{\ell, \mathfrak{p}}$  into a representation  $\pi_{\ell, \mathfrak{p}}$  on the space  $L^2(\mathfrak{s})$ :

$$\pi_{\ell, \mathfrak{p}} := U_\ell^* \circ \sigma_{\ell, \mathfrak{p}} \circ U_\ell. \quad (1)$$

Furthermore, one can express the representation  $\sigma_{\ell, \mathfrak{p}}$  in the following way:

$$\begin{aligned} \sigma_{\ell, \mathfrak{p}}(S \cdot Y)\xi(R) &= \xi(Y^{-1}S^{-1}R) \\ &= \xi\left((R-S) \cdot \left(-Y + \frac{1}{2}[R, S] - \frac{1}{2}[R-S, Y]\right)\right) \\ &= e^{2\pi i \langle \ell, -Y + \frac{1}{2}[R, S] - \frac{1}{2}[R-S, Y] \rangle} \xi(R-S) \\ &\quad \forall R, S \in \mathfrak{s} \quad \forall Y \in \mathfrak{p} \quad \forall \xi \in L^2(G/P, \chi_\ell). \end{aligned}$$

Hence

$$\begin{aligned} \pi_{\ell, \mathfrak{p}}(S \cdot Y)\varphi(R) &= e^{2\pi i \langle \ell, -Y + \frac{1}{2}[R, S] - \frac{1}{2}[R-S, Y] \rangle} \varphi(R-S) \\ &\quad \forall R, S \in \mathfrak{s} \quad \forall Y \in \mathfrak{p} \quad \forall \varphi \in L^2(\mathfrak{s}). \end{aligned} \quad (2)$$

### 2.3 Orbit method

By the **Kirillov theory** (see [2], Section 2.2), for every representation class  $\gamma \in \widehat{G}$ , there exists an element  $\ell \in \mathfrak{g}^*$  and a polarization  $\mathfrak{p}$  of  $\ell$  in  $\mathfrak{g}$  such that  $\gamma = [\text{ind}_P^G \chi_\ell]$ , whereat  $P := \exp(\mathfrak{p})$ . Moreover, if  $\ell, \ell' \in \mathfrak{g}^*$  are located in the same coadjoint orbit  $O \in \mathfrak{g}^*/G$  and  $\mathfrak{p}$  and  $\mathfrak{p}'$  are polarizations in  $\ell$  and  $\ell'$ , respectively, the induced representations  $\text{ind}_P^G \chi_\ell$  and  $\text{ind}_{P'}^G \chi_{\ell'}$  are equivalent and thus, the Kirillov map which goes from the coadjoint orbit space  $\mathfrak{g}^*/G$  to the spectrum  $\widehat{G}$  of  $G$

$$K : \mathfrak{g}^*/G \rightarrow \widehat{G}, \quad \text{Ad}^*(G)\ell \mapsto [\text{ind}_P^G \chi_\ell]$$

is a homeomorphism (see [1] or [5], Chapter 3). Therefore,

$$\mathfrak{g}^*/G \cong \widehat{G}$$

as topological spaces.

For every  $\ell \in \mathfrak{g}^*$  and  $x \in G = (\mathfrak{g}, \cdot)$

$$\text{Ad}^*(x)\ell = (\mathbb{I}_{\mathfrak{g}^*} + \text{ad}^*(x))\ell \in \ell + \mathfrak{g}(\ell)^\perp.$$

Hence, as  $\text{ad}^*(\mathfrak{g})\ell = \mathfrak{g}(\ell)^\perp$ ,

$$O_\ell := \text{Ad}^*(G)\ell = \ell + \mathfrak{g}(\ell)^\perp. \quad (3)$$

#### Definition 2.2.

Let  $T$  be a second countable topological space and suppose that  $T$  is not Hausdorff, which means that converging sequences can have many limit points. Denote by  $L((t_k)_k)$  the collection of all the limit points of a sequence  $(t_k)_k$  in  $T$ . A sequence  $(t_k)_k$  is called *properly converging*, if  $(t_k)_k$  has limit points and if every subsequence of  $(t_k)_k$  has the same limit set as  $(t_k)_k$ .

It is well known that every converging sequence in  $T$  admits a properly converging subsequence.

Now, let  $(\pi_k)_k \subset \widehat{G}$  be a properly converging sequence in  $\widehat{G}$  with limit set  $L((\pi_k)_k)$ . Let  $O \in \mathfrak{g}^*/G$  be the Kirillov orbit of some  $\pi \in L((\pi_k)_k)$ ,  $O_k$  the Kirillov orbit of  $\pi_k$  for every  $k$  and let  $\ell \in O$ . Then there exists for every  $k$  an element  $\ell_k \in O_k$ , such that  $\lim_{k \rightarrow \infty} \ell_k = \ell$  in  $\mathfrak{g}^*$  (see [5]). One can assume that, passing to a subsequence if necessary, the sequence  $(\mathfrak{g}(\ell_k))_k$  converges in the subspace topology to a subalgebra  $\mathfrak{u}$  of  $\mathfrak{g}(\ell)$  and that there exists a number  $d \in \mathbb{N}$ , such that  $\dim(O_k) = d$  for every  $k \in \mathbb{N}$ . Then it follows from (3), that

$$L((O_k)_k) = \lim_{k \rightarrow \infty} \ell_k + \mathfrak{g}(\ell_k)^\perp = \ell + \mathfrak{u}^\perp \subset \mathfrak{g}^*. \quad (4)$$

Since  $\mathfrak{g}(\ell_k)$  contains  $[\mathfrak{g}, \mathfrak{g}]$  for every  $k$ , the subspace  $\mathfrak{u}$  also contains  $[\mathfrak{g}, \mathfrak{g}]$ . Hence, the limit set  $L((\pi_k)_k)$  in  $\widehat{G}$  of the sequence  $(\pi_k)_k$  is the "affine" subset

$$L((\pi_k)_k) = \{ [\chi_q \otimes \text{ind}_P^G \chi_\ell] \mid q \in \mathfrak{u}^\perp \}$$

for a polarization  $\mathfrak{p}$  in  $\ell$  and  $P := \exp(\mathfrak{p})$ .

The observations above lead to the following proposition:

**Proposition 2.3.**

*There are three different types of possible limit sets of the sequence  $(O_k)_k$  of coadjoint orbits:*

1. *The limit set  $L((O_k)_k)$  is the singleton  $O_\ell = \ell + \mathfrak{g}(\ell)^\perp$ , i.e.  $\mathfrak{u} = \mathfrak{g}(\ell)$ .*
2. *The limit set  $L((O_k)_k)$  is the affine subspace  $\ell + \mathfrak{u}^\perp$  of characters of  $\mathfrak{g}$ , i.e.  $\langle \ell, [\mathfrak{g}, \mathfrak{g}] \rangle = \{0\}$ .*
3. *The dimension of the orbit  $O_\ell$  is strictly greater than 0 and strictly smaller than  $d$ . In this case*

$$L((O_k)_k) = \bigcup_{q \in \mathfrak{u}^\perp} q + O_\ell, \quad \text{i.e.} \quad L((\pi_k)_k) = \bigcup_{q \in \mathfrak{u}^\perp} [\chi_q \otimes \text{ind}_P^G \chi_\ell]$$

*for a polarization  $\mathfrak{p}$  in  $\ell$  and  $P := \exp(\mathfrak{p})$ .*

## 2.4 The $C^*$ -algebra $C^*(G/U, \chi_\ell)$

Let  $\mathfrak{u} \subset \mathfrak{g}$  be an ideal of  $\mathfrak{g}$  containing  $[\mathfrak{g}, \mathfrak{g}]$ ,  $U := \exp(\mathfrak{u})$  and let  $\ell \in \mathfrak{g}^*$  such that  $\langle \ell, [\mathfrak{g}, \mathfrak{u}] \rangle = \{0\}$  and such that  $\mathfrak{u} \subset \mathfrak{g}(\ell)$ . Then the character  $\chi_\ell$  of the group  $U = \exp(\mathfrak{u})$  is  $G$ -invariant. One can thus define the involutive Banach algebra  $L^1(G/U, \chi_\ell)$  as

$$\begin{aligned} L^1(G/U, \chi_\ell) &:= \left\{ f : G \rightarrow \mathbb{C} \mid f \text{ measurable, } f(gu) = \chi_\ell(u^{-1})f(g) \ \forall g \in G \right. \\ &\quad \left. \forall u \in U, \|f\|_1 := \int_{G/U} |f(g)| \, d\dot{g} < \infty \right\}. \end{aligned}$$

The convolution

$$f * f'(g) := \int_{G/U} f(x) f'(x^{-1}g) \, d\dot{x} \quad \forall g \in G$$

and the involution

$$f^*(g) := \overline{f(g^{-1})} \quad \forall g \in G$$

are well-defined for  $f, f' \in L^1(G/U, \chi_\ell)$  and

$$\|f * f'\|_1 \leq \|f\|_1 \|f'\|_1.$$

Moreover, the linear mapping

$$\begin{aligned} p_{G/U} &: L^1(G) \rightarrow L^1(G/U, \chi_\ell), \\ p_{G/U}(F)(g) &:= \int_U F(gu) \chi_\ell(u) \, du \quad \forall F \in L^1(G) \quad \forall g \in G \end{aligned}$$

is a surjective  $*$ -homomorphism between the algebras  $L^1(G)$  and  $L^1(G/U, \chi_\ell)$ .  
Let

$$\widehat{G}_{\mathfrak{u}, \ell} := \left\{ (\pi, \mathcal{H}_\pi) \in \widehat{G} \mid \pi|_U = \chi_\ell|_U \mathbb{I}_{\mathcal{H}_\pi} \right\}.$$

Then  $\widehat{G}_{\mathfrak{u}, \ell}$  is a closed subset of  $\widehat{G}$ , which can be identified with the spectrum of the algebra  $L^1(G/U, \chi_\ell)$ . Indeed it is easy to see that every irreducible unitary representation  $(\pi, \mathcal{H}_\pi) \in \widehat{G}_{\mathfrak{u}, \ell}$  defines an irreducible representation  $(\tilde{\pi}, \mathcal{H}_\pi)$  of the algebra  $L^1(G/U, \chi_\ell)$  as follows:

$$\tilde{\pi}(p_{G/U}(F)) := \pi(F) \quad \forall F \in L^1(G).$$

Similarly, if  $(\tilde{\pi}, \mathcal{H}_{\tilde{\pi}})$  is an irreducible unitary representation of  $L^1(G/U, \chi_\ell)$  then

$$\pi := \tilde{\pi} \circ p_{G/U}$$

defines an element of  $\widehat{G}_{\mathfrak{u}, \ell}$ .

Let  $\mathfrak{s} \subset \mathfrak{g}$  be a subspace of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{g}(\ell) \oplus \mathfrak{s}$ . Since  $\mathfrak{u}$  contains  $[\mathfrak{g}, \mathfrak{g}]$ , it is easy to see that

$$\widehat{G}_{\mathfrak{u}, \ell} = \left\{ [\chi_q \otimes \pi_\ell] \mid q \in (\mathfrak{u} + \mathfrak{s})^\perp \right\},$$

letting  $\pi_\ell := \text{ind}_P^G \chi_\ell$  for a polarization  $\mathfrak{p}$  in  $\ell$  and  $P := \exp(\mathfrak{p})$ .

Denote by  $C^*(G/U, \chi_\ell)$  the  $C^*$ -algebra of  $L^1(G/U, \chi_\ell)$ , whose spectrum can also be identified with  $\widehat{G}_{\mathfrak{u}, \ell}$ .

With  $\pi_{\ell+q} = \text{ind}_P^G \chi_{\ell+q}$ , the Fourier transform  $\mathcal{F}$  defined by

$$\mathcal{F}(a)(q) := \pi_{\ell+q}(a) \quad \forall q \in (\mathfrak{u} + \mathfrak{s})^\perp$$

then maps the  $C^*$ -algebra  $C^*(G/U, \chi_\ell)$  onto the algebra  $C_0((\mathfrak{u} + \mathfrak{s})^\perp, \mathcal{K}(\mathcal{H}_{\pi_\ell}))$  of the continuous mappings  $\varphi : (\mathfrak{u} + \mathfrak{s})^\perp \rightarrow \mathcal{K}(\mathcal{H}_{\pi_\ell})$  vanishing at infinity with values in the algebra of compact operators on the Hilbert space of the representation  $\pi_\ell$ .

If one restricts  $p_{G/U}$  to the Fréchet algebra  $\mathcal{S}(G) \subset L^1(G)$ , its image will be the Fréchet algebra

$$\begin{aligned} \mathcal{S}(G/U, \chi_\ell) &= \left\{ f \in L^1(G/U, \chi_\ell) \mid f \text{ smooth and for every subspace } \mathfrak{s}' \subset \mathfrak{g} \text{ with} \right. \\ &\quad \left. \mathfrak{g} = \mathfrak{s}' \oplus \mathfrak{u} \text{ and for } S' = \exp(\mathfrak{s}'), f|_{S'} \in \mathcal{S}(S') \right\}. \end{aligned}$$

### 3 Conditions 1, 2 and 3(a)

Now, to start with the proof of the above listed conditions, the families of sets  $(S_i)_{i \in \{0, \dots, r\}}$  and  $(\Gamma_i)_{i \in \{0, \dots, r\}}$  are going to be defined and the Properties 1, 2 and 3(a) are going to be checked. In order to be able to do this, one needs to construct a polarization  $\mathfrak{p}_\ell^V$  for  $\ell \in \mathfrak{g}^*$  as follows:

Fix once and for all a Jordan-Hölder basis  $\{H_1, \dots, H_n\}$  of  $\mathfrak{g}$ , in such a way that  $\mathfrak{g}_i := \text{span}\{H_i, \dots, H_n\}$  for  $i \in \{0, \dots, n\}$  is an ideal in  $\mathfrak{g}$ . Since  $\mathfrak{g}$  is two-step nilpotent, one can first choose a basis  $\{H_{\tilde{n}}, \dots, H_n\}$  of  $[\mathfrak{g}, \mathfrak{g}]$  and then add the vectors  $H_1, \dots, H_{\tilde{n}-1}$  to obtain a basis of  $\mathfrak{g}$ . Let

$$I_\ell^{Puk} := \{i \leq n \mid \mathfrak{g}(\ell) \cap \mathfrak{g}_i = \mathfrak{g}(\ell) \cap \mathfrak{g}_{i+1}\}$$

be the Pukanszky index set for  $\ell \in \mathfrak{g}^*$ . The number of elements  $|I_\ell^{Puk}|$  of  $I_\ell^{Puk}$  is the dimension of the orbit  $O_\ell$  of  $\ell$ .

Moreover, if one denotes by  $\mathfrak{g}_i(\ell|_{\mathfrak{g}_i})$  the stabilizer of  $\ell|_{\mathfrak{g}_i}$  in  $\mathfrak{g}_i$ ,

$$\mathfrak{p}_\ell^V := \sum_{i=1}^n \mathfrak{g}_i(\ell|_{\mathfrak{g}_i})$$

is the Vergne polarization of  $\ell$  in  $\mathfrak{g}$ . Its construction will now be analyzed by a method developed in [8], Section 1:

Let  $\ell \in \mathfrak{g}^*$ . Then choose the greatest index  $j_1(\ell) \in \{1, \dots, n\}$  such that  $H_{j_1(\ell)} \notin \mathfrak{g}(\ell)$  and let  $Y_1^{V,\ell} := H_{j_1(\ell)}$ . Furthermore, choose the index  $k_1(\ell) \in \{1, \dots, n\}$  such that  $\langle \ell, [H_{k_1(\ell)}, H_{j_1(\ell)}] \rangle \neq 0$  and  $\langle \ell, [H_i, H_{j_1(\ell)}] \rangle = 0$  for all  $i > k_1(\ell)$  and let  $X_1^{V,\ell} := H_{k_1(\ell)}$ .

Next, let  $\mathfrak{g}^{1,\ell} := \{U \in \mathfrak{g} \mid \langle \ell, [U, Y_1^{V,\ell}] \rangle = 0\}$ . Then  $\mathfrak{g}^{1,\ell}$  is an ideal in  $\mathfrak{g}$  which does not contain  $X_1^{V,\ell}$ , and  $\mathfrak{g} = \mathbb{R}X_1^{V,\ell} \oplus \mathfrak{g}^{1,\ell}$ . Now, the Jordan-Hölder basis will be changed, taking out  $H_{k_1(\ell)}$ :

Consider the Jordan-Hölder basis  $\{H_1^{1,\ell}, \dots, H_{k_1(\ell)-1}^{1,\ell}, H_{k_1(\ell)+1}^{1,\ell}, \dots, H_n^{1,\ell}\}$  of  $\mathfrak{g}^{1,\ell}$  with

$$H_i^{1,\ell} := H_i \quad \forall i > k_1(\ell) \quad \text{and} \quad H_i^{1,\ell} := H_i - \frac{\langle \ell, [H_i, Y_1^{V,\ell}] \rangle X_1^{V,\ell}}{\langle \ell, [X_1^{V,\ell}, Y_1^{V,\ell}] \rangle} \quad \forall i < k_1(\ell).$$

Then, choose the greatest index  $j_2(\ell) \in \{1, \dots, k_1(\ell) - 1, k_1(\ell) + 1, \dots, n\}$  in such a way that  $H_{j_2(\ell)}^{1,\ell} \notin \mathfrak{g}^{1,\ell}(\ell|_{\mathfrak{g}^{1,\ell}})$  and define  $Y_2^{V,\ell} := H_{j_2(\ell)}^{1,\ell}$ . Like above, choose  $k_2(\ell) \in \{1, \dots, k_1(\ell) - 1, k_1(\ell) + 1, \dots, n\}$  such that  $\langle \ell, [H_{k_2(\ell)}^{1,\ell}, H_{j_2(\ell)}^{1,\ell}] \rangle \neq 0$  and that  $\langle \ell, [H_i^{1,\ell}, H_{j_2(\ell)}^{1,\ell}] \rangle = 0$  for all  $i > k_2(\ell)$  and set  $X_2^{V,\ell} := H_{k_2(\ell)}^{1,\ell}$ .

Iterating this procedure, one gets sets  $\{Y_1^{V,\ell}, \dots, Y_d^{V,\ell}\}$  and  $\{X_1^{V,\ell}, \dots, X_d^{V,\ell}\}$  for  $d \in \{0, \dots, [\frac{n}{2}]\}$  with the properties

$$\mathfrak{p}_\ell^V = \text{span}\{Y_1^{V,\ell}, \dots, Y_d^{V,\ell}\} \oplus \mathfrak{g}(\ell)$$

and

$$\begin{aligned} \langle \ell, [X_i^{V,\ell}, Y_j^{V,\ell}] \rangle &\neq 0, \quad \langle \ell, [X_i^{V,\ell}, Y_j^{V,\ell}] \rangle = 0 \quad \forall i \neq j \in \{1, \dots, d\} \quad \text{and} \\ \langle \ell, [Y_i^{V,\ell}, Y_j^{V,\ell}] \rangle &= 0 \quad \forall i, j \in \{1, \dots, d\}. \end{aligned}$$

Now, let

$$J(\ell) := \{j_1(\ell), \dots, j_d(\ell)\} \quad \text{and} \quad K(\ell) := \{k_1(\ell), \dots, k_d(\ell)\}.$$

Then

$$I_\ell^{Puk} = J(\ell) \dot{\cup} K(\ell) \quad \text{and} \quad j_1(\ell) > \dots > j_d(\ell).$$

It is easy to see that the index sets  $I_\ell^{Puk}$ ,  $J(\ell)$  and  $K(\ell)$  are the same on every coadjoint orbit (see [8]) and can therefore also be denoted by  $I_O^{Puk}$ ,  $J(O)$  and  $K(O)$  if  $\ell$  is located in the coadjoint orbit  $O$ .

Now, for the parametrization of  $\mathfrak{g}^*/G$  and thus of  $\widehat{G}$  and for the choice of the in Property 1(b) required concrete realization of a representation, let  $O \in \mathfrak{g}^*/G$ . A theorem of L.Pukanszky (see [9], Part II, Chapter I.3 or [8], Corollary 1.2.5) states that there exists one unique  $\ell_O \in O$  such that  $\ell_O(H_i) = 0$  for every index  $i \in I_O^{Puk}$ . So, choose this  $\ell_O$ , let  $P_{\ell_O}^V := \exp(\mathfrak{p}_{\ell_O}^V)$  and define the irreducible unitary representation

$$\pi_{\ell_O}^V := \text{ind}_{P_{\ell_O}^V}^G \chi_{\ell_O}$$

associated to the orbit  $O$  and acting on  $L^2(G/P_{\ell_O}^V, \chi_{\ell_O}) \cong L^2(\mathbb{R}^d)$ .

Next, one has to construct the demanded sets  $\Gamma_i$  for  $i \in \{0, \dots, r\}$ :

For this, define for a pair of sets  $(J, K)$  such that  $J, K \subset \{1, \dots, n\}$ ,  $|J| = |K|$  and  $J \cap K = \emptyset$  the subset  $(\mathfrak{g}^*/G)_{(J,K)}$  of  $\mathfrak{g}^*/G$  by

$$(\mathfrak{g}^*/G)_{(J,K)} := \{O \in \mathfrak{g}^*/G \mid (J, K) = (J(O), K(O))\}.$$

Moreover, let

$$\mathcal{M} := \{(J, K) \mid J, K \subset \{1, \dots, n\}, J \cap K = \emptyset, |J| = |K|, (\mathfrak{g}^*/G)_{(J,K)} \neq \emptyset\}$$

and

$$(\mathfrak{g}^*/G)_{2d} := \{O \in \mathfrak{g}^*/G \mid |I_O^{Puk}| = 2d\}.$$

Then

$$(\mathfrak{g}^*/G)_{2d} = \bigcup_{\substack{(J,K): J,K \subset \{1,\dots,n\}, \\ |J|=|K|=d, J \cap K = \emptyset}} (\mathfrak{g}^*/G)_{(J,K)}$$

and

$$\mathfrak{g}^*/G = \bigcup_{d \in \{0, \dots, [\frac{n}{2}]\}} (\mathfrak{g}^*/G)_{2d} = \bigcup_{(J,K) \in \mathcal{M}} (\mathfrak{g}^*/G)_{(J,K)}.$$

Now, an order on the set  $\mathcal{M}$  shall be introduced.

First, if  $|J| = |K| = d$ ,  $|J'| = |K'| = d'$  and  $d < d'$ , then the pair  $(J, K)$  is defined to be smaller than the pair  $(J', K')$ :  $(J, K) < (J', K')$ .

If  $|J| = |K| = |J'| = |K'| = d$ ,  $J = \{j_1, \dots, j_d\}$ ,  $J' = \{j'_1, \dots, j'_d\}$  and  $j_1 < j'_1$ , the pair  $(J, K)$  is again defined to be smaller than  $(J', K')$ .

Otherwise, if  $j_1 = j'_1$ , one has to consider  $K = \{k_1, \dots, k_d\}$  and  $K' = \{k'_1, \dots, k'_d\}$  and here again, compare the first elements  $k_1$  and  $k'_1$ : So, if  $j_1 = j'_1$  and  $k_1 < k'_1$ , again  $(J, K) < (J', K')$ .

But if  $k_1 = k'_1$ , one compares  $j_2$  and  $j'_2$  and continues in that way.

If  $r+1 = |\mathcal{M}|$ , one can identify the ordered set  $\mathcal{M}$  with the interval  $\{0, \dots, r\}$  and assign to each such pair  $(J, K) \in \mathcal{M}$  a number  $i_{JK} \in \{0, \dots, r\}$ .

Finally, one can therefore define the sets  $\Gamma_{i_{JK}}$  and  $S_{i_{JK}}$  as

$$\Gamma_{i_{JK}} := \{[\pi_{\ell_O}^V] \mid O \in (\mathfrak{g}^*/G)_{(J,K)}\} \quad \text{and}$$

$$S_{i_{JK}} := \bigcup_{i \in \{0, \dots, i_{JK}\}} \Gamma_i.$$

Then obviously, the family  $(S_i)_{i \in \{0, \dots, r\}}$  is an increasing family in  $\widehat{G}$ .

Furthermore, the set  $S_i$  is closed for every  $i \in \{0, \dots, r\}$ . This can easily be deduced from the definition of the index sets  $J(\ell)$  and  $K(\ell)$ . The indices  $j_m(\ell)$  and  $k_m(\ell)$  for  $m \in \{1, \dots, d\}$  are chosen in such a way that they are the largest to fulfill a condition of the type  $\langle \ell, [H_{j_m(\ell)}^{m-1, \ell}, \cdot] \rangle \neq 0$  or  $\langle \ell, [H_{k_m(\ell)}^{m-1, \ell}, \cdot] \rangle \neq 0$ , respectively.

In addition, the sets  $\Gamma_i$  are Hausdorff. For this, let  $i = i_{JK}$  for  $(J, K) \in \mathcal{M}$  and  $(O_k)_k$  in  $(\mathfrak{g}^*/G)_{(J,K)}$  a sequence of orbits such that the sequence  $([\pi_{\ell_{O_k}}^V])_k$  converges in  $\Gamma_i$ , i.e.  $(O_k)_k$  converges in  $(\mathfrak{g}^*/G)_{(J,K)}$  and thus has a limit point  $O$  in  $(\mathfrak{g}^*/G)_{(J,K)}$ . If now  $O_k \ni \ell_k \xrightarrow{k \rightarrow \infty} \ell \in O$ , then by (4), it follows that the limit  $\mathfrak{u}$  of the sequence  $(\mathfrak{g}(\ell_k))_k$  is equal to  $\mathfrak{g}(\ell)$ . Therefore, the sequence  $(O_k)_k$  and thus also the sequence  $([\pi_{\ell_{O_k}}^V])_k$  have unique limits and hence  $\Gamma_i$  is Hausdorff.

Moreover, one can still observe that for  $d = 0$  the choice  $J = K = \emptyset$  represents the only possibility to get  $|J| = |K| = d$ . So, the pair  $(\emptyset, \emptyset)$  is the first element in the above defined order and therefore corresponds to 0. Thus

$$\Gamma_0 = \{[\pi_{\ell_O}^V] \mid I_O^{Puk} = \emptyset\},$$

which is equivalent to the fact that  $\mathfrak{g}(\ell_O) = \mathfrak{g}$  which again is equivalent to the fact that every  $\pi_{\ell_O}^V \in \Gamma_0$  is a character. Hence,  $S_0 = \Gamma_0$  is the set of all characters on  $\mathfrak{g}$ , as demanded.



Since one can identify the quotient space  $G/P_{\ell_O}^V$  with  $\mathbb{R}^d$  by means of the subspace  $\mathfrak{s}_{\ell_O} = \text{span}\{X_1^{V,\ell_O}, \dots, X_d^{V,\ell_O}\}$ , one can identify the Hilbert space  $L^2(G/P_{\ell_O}^V, \chi_{\ell_O})$  with  $L^2(\mathbb{R}^d)$  as in (1) and one can suppose that the representation  $\pi_{\ell_O}^V$  acts on the Hilbert space  $L^2(\mathbb{R}^d)$  for every  $O \in (\mathfrak{g}^*/G)_{2d}$ .

Hence, the first condition is fulfilled. For the proof of the Properties 2 and 3(a), a proposition will be shown:

**Proposition 3.1.**

For every  $a \in C^*(G)$  and every  $(J, K) \in \mathcal{M}$  with  $|J| = |K| = d \in \{0, \dots, [\frac{n}{2}]\}$ , the mapping

$$\Gamma_{i_{JK}} \rightarrow L^2(\mathbb{R}^d), \gamma \mapsto \mathcal{F}(a)(\gamma)$$

is norm continuous and the operator  $\mathcal{F}(a)(\gamma)$  is compact for all  $\gamma \in \Gamma_{i_{JK}}$ .

Proof:

The compactness follows directly from a general theorem which can be found in [2] (Chapter 4.2) or [9] (Part II, Chapter II.5) and states that the  $C^*$ -algebra  $C^*(G)$  of every connected nilpotent Lie group  $G$  fulfills the CCR condition, i.e. the image of every irreducible representation of  $C^*(G)$  is a compact operator.

Next, let  $d \in \{0, \dots, [\frac{n}{2}]\}$  and  $(J, K) \in \mathcal{M}$  such that  $|J| = |K| = d$ .

First, one has to observe that the polarization  $\mathfrak{p}_{\ell}^V$  is continuous in  $\ell$  on the set  $\{\ell_{O'} \mid O' \in (\mathfrak{g}^*/G)_{(J,K)}\}$ . This can be seen by the construction of the vectors  $\{Y_1^{V,\ell}, \dots, Y_d^{V,\ell}\}$ .

Now, let  $(O_k)_k$  be a sequence in  $(\mathfrak{g}^*/G)_{(J,K)}$  and  $O \in (\mathfrak{g}^*/G)_{(J,K)}$  such that  $[\pi_{\ell_{O_k}}^V] \xrightarrow{k \rightarrow \infty} [\pi_{\ell_O}^V]$  and let  $a \in C^*(G)$ . Then  $\ell_{O_k} \xrightarrow{k \rightarrow \infty} \ell_O$  and by the observation above, the associated sequence of polarizations  $(\mathfrak{p}_{\ell_{O_k}}^V)_k$  converges to the polarization  $\mathfrak{p}_{\ell_O}^V$ . By Theorem 2.3 in [11], thus  $\pi_{\ell_{O_k}}^V(a) \xrightarrow{k \rightarrow \infty} \pi_{\ell_O}^V(a)$  in the operator norm.

□

Since  $C^*(G)$  is obviously separable, this proposition proves the desired Properties 2 and 3(a) and hence, it remains to show Property 3(b):

## 4 Condition 3(b)

### 4.1 Introduction to the setting

For simplicity, in the following, the representations will be identified with their equivalence classes.

Let  $d \in \{0, \dots, [\frac{n}{2}]\}$  and  $(J, K) \in \mathcal{M}$  with  $|J| = |K| = d$ . Furthermore, fix  $i = i_{JK} \in \{0, \dots, r\}$ .

Let  $(\pi_k^V)_k = (\pi_{\ell_{O_k}}^V)_k$  be a sequence in  $\Gamma_i$  whose limit set is located outside  $\Gamma_i$ . Since every converging sequence has a properly converging subsequence, it will be assumed that  $(\pi_k^V)_k$  is properly converging and the transition to a subsequence will be omitted.

The corresponding sequence of coadjoint orbits  $(O_k)_k$  is contained in  $(\mathfrak{g}^*/G)_{(J,K)}$  and in particular every  $O_k$  has the same dimension  $2d$ . Moreover, it converges properly to a set of orbits  $L((O_k)_k)$ . In addition, since  $S_i$  is closed, the limit set  $L((\pi_k^V)_k)$  of the sequence  $(\pi_k^V)_k$  is contained in  $S_{i-1} = \bigcup_{j \in \{0, \dots, i-1\}} \Gamma_j$  and therefore for every element  $O \in L((O_k)_k)$  there exists a pair

$(J_O, K_O) < (J, K)$  such that  $\pi_{\ell_O}^V \in \Gamma_{i_{J_O K_O}}$  or equivalently,  $O \in (\mathfrak{g}^*/G)_{(J_O, K_O)}$ .

## 4.2 Changing the Jordan-Hölder basis.

Let  $\tilde{\ell} \in \tilde{O} \in L((O_k)_k)$ . Then, there exists a sequence  $(\tilde{\ell}_k)_k$  in  $O_k$  such that  $\tilde{\ell} = \lim_{k \rightarrow \infty} \tilde{\ell}_k$ .

Since one is interested in the orbits  $O_k = \tilde{\ell}_k + \mathfrak{g}(\tilde{\ell}_k)^\perp$ , one can change the sequence  $(\tilde{\ell}_k)_k$  to a sequence  $(\ell_k)_k$  by letting  $\ell_k(A) = 0$  for every  $A \in \mathfrak{g}(\tilde{\ell}_k)^\perp = \mathfrak{g}(\ell_k)^\perp$ .

Thus, one obtains another converging sequence  $(\ell_k)_k$  in  $(O_k)_k$  whose limit  $\ell$  is located in an orbit  $O \in L((O_k)_k)$ .

As above, one can suppose that the subalgebras  $(\mathfrak{g}(\ell_k))_k$  converge to a subalgebra  $\mathfrak{u}$ , whose corresponding Lie group  $\exp(\mathfrak{u})$  is denoted by  $U$ . These subalgebras  $\mathfrak{g}(\ell_k)$  can be written as

$$\mathfrak{g}(\ell_k) = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{s}_k,$$

where  $\mathfrak{s}_k \subset [\mathfrak{g}, \mathfrak{g}]^\perp$ . In addition, let  $\mathfrak{n}_{k,0}$  be the kernel of  $\ell_k|_{[\mathfrak{g}, \mathfrak{g}]}$  and  $\mathfrak{s}_{k,0}$  the kernel of  $\ell_k|_{\mathfrak{s}_k}$  for all  $k \in \mathbb{N}$ . One can assume that  $\mathfrak{s}_{k,0} \neq \mathfrak{s}_k$  and choose  $T_k \in \mathfrak{s}_k$  orthogonal to  $\mathfrak{s}_{k,0}$  of length 1. The case  $\mathfrak{s}_{k,0} = \mathfrak{s}_k$  for  $k \in \mathbb{N}$ , being easier, will be omitted.

Similarly, choose  $Z_k \in [\mathfrak{g}, \mathfrak{g}]$  orthogonal to  $\mathfrak{n}_{k,0}$  of length 1. One sees that such a  $Z_k$  must exist: If  $\ell_k|_{[\mathfrak{g}, \mathfrak{g}]} = 0$  for  $k \in \mathbb{N}$ , then  $\pi_{\ell_{O_k}}^V$  is a character and thus contained in  $S_0 = \Gamma_0$ . But  $S_0 = \Gamma_0$  is closed and thus  $(\pi_{\ell_{O_k}}^V)_k$  cannot have a limit set outside  $\Gamma_0$ .

Furthermore, let  $\mathfrak{r}_k = \mathfrak{g}(\ell_k)^\perp \subset \mathfrak{g}$ .

One can assume that, passing to a subsequence if necessary,  $\lim_{k \rightarrow \infty} Z_k =: Z$ ,  $\lim_{k \rightarrow \infty} T_k =: T$  and  $\lim_{k \rightarrow \infty} \mathfrak{r}_k =: \mathfrak{r}$  exist.

Now, new polarizations  $\mathfrak{p}_k$  in  $\ell_k$  are needed:

The restriction to  $\mathfrak{r}_k$  of the skew-form  $B_k := B_{\ell_k}$  defined in Chapter 2 is non-degenerate on  $\mathfrak{r}_k$  and there exists an invertible endomorphism  $S_k$  of  $\mathfrak{r}_k$  such that

$$\langle x, S_k(x') \rangle = B_k(x, x') \quad \forall x, x' \in \mathfrak{r}_k.$$

Then  $S_k$  is skew-symmetric, i.e.  $S_k^t = -S_k$ , and with the help of Lemma 6.1 one can decompose  $\mathfrak{r}_k$  into an orthogonal direct sum

$$\mathfrak{r}_k = \sum_{j=1}^d V_k^j$$

of two-dimensional  $S_k$ -invariant subspaces. Choose an orthonormal basis  $\{X_j^k, Y_j^k\}$  of  $V_k^j$ . Then,

$$\begin{aligned} [X_i^k, X_j^k] &\in \mathfrak{n}_{k,0} \quad \forall i, j \in \{1, \dots, d\}, \\ [Y_i^k, Y_j^k] &\in \mathfrak{n}_{k,0} \quad \forall i, j \in \{1, \dots, d\} \quad \text{and} \\ [X_i^k, Y_j^k] &= \delta_{i,j} c_j^k Z_k \mod \mathfrak{n}_{k,0} \quad \forall i, j \in \{1, \dots, d\}, \end{aligned}$$

whereat  $0 \neq c_j^k \in \mathbb{R}$  and  $\sup_{k \in \mathbb{N}} c_j^k < \infty$  for every  $j \in \{1, \dots, d\}$ .

Again, by passing to a subsequence if necessary, the sequence  $(c_j^k)_k$  converges for every  $j \in \{1, \dots, d\}$  to some  $c_j$ .

Since  $X_j^k, Y_j^k \in \mathfrak{r}_k$  and  $\ell_k(A) = 0$  for every  $A \in \mathfrak{r}_k$ ,  $\ell_k(X_j^k) = \ell_k(Y_j^k) = 0$  for all  $j \in \{1, \dots, d\}$ . Furthermore, one can suppose that the sequences  $(X_j^k)_k, (Y_j^k)_k$  converge in  $\mathfrak{g}$  to vectors  $X_j, Y_j$  which form a basis modulo  $\mathfrak{u}$  in  $\mathfrak{g}$ .

It follows that

$$\begin{aligned} \langle \ell_k, [X_j^k, Y_j^k] \rangle &= c_j^k \lambda_k, \quad \text{where} \\ \lambda_k &= \langle \ell_k, Z_k \rangle \xrightarrow{k \rightarrow \infty} \langle \ell, Z \rangle =: \lambda. \end{aligned}$$

As  $Z_k$  was chosen orthogonal to  $\mathfrak{n}_{k,0}$ ,  $\lambda_k \neq 0$  for every  $k$ .

Now, let

$$\mathfrak{p}_k := \text{span}\{Y_1^k, \dots, Y_d^k, \mathfrak{g}(\ell_k)\}$$

and  $P_k := \exp(\mathfrak{p}_k)$ . Then  $\mathfrak{p}_k$  is a polarization at  $\ell_k$ . Furthermore, define the representation  $\pi_k$  as

$$\pi_k := \text{ind}_{P_k}^G \chi_{\ell_k}.$$

Then, since  $\pi_k$ , as well as  $\pi_k^V$  are induced representations of polarizations and of the characters  $\chi_{\ell_k}$  and  $\chi_{\ell_{O_k}}$ , whereat  $\ell_k$  and  $\ell_{O_k}$  lie in the same coadjoint orbit  $O_k$ , the two representations are equivalent, as observed in Chapter 2.

Let  $\mathfrak{a}_k := \mathfrak{n}_{k,0} + \mathfrak{s}_{k,0}$ . Then  $\mathfrak{a}_k$  is an ideal of  $\mathfrak{g}$  on which  $\ell_k$  is 0. Therefore, the normal subgroup  $\exp(\mathfrak{a}_k)$  is contained in the kernel of the representation  $\pi_k$ . Moreover, let  $\mathfrak{a} := \lim_{k \rightarrow \infty} \mathfrak{a}_k$ .

In addition, let  $p \in \mathbb{N}$ ,  $\tilde{p} \in \{1, \dots, p\}$  and let  $\{A_1^k, \dots, A_{\tilde{p}}^k\}$  denote an orthonormal basis of  $\mathfrak{n}_{k,0}$ , the part of  $\mathfrak{a}_k$  which lies inside  $[\mathfrak{g}, \mathfrak{g}]$ , and  $\{A_{\tilde{p}+1}^k, \dots, A_p^k\}$  an orthonormal basis of  $\mathfrak{s}_{k,0}$ , the part of  $\mathfrak{a}_k$  outside  $[\mathfrak{g}, \mathfrak{g}]$ . Then  $\{A_1^k, \dots, A_p^k\}$  is an orthonormal basis of  $\mathfrak{a}_k$  and as above, one can assume that  $\lim_{k \rightarrow \infty} A_j^k = A_j$  exists for all  $j \in \{1, \dots, p\}$ .

Now, for every  $k \in \mathbb{N}$  one can take as an orthonormal basis for  $\mathfrak{g}$  the set of vectors

$$\{X_1^k, \dots, X_d^k, Y_1^k, \dots, Y_d^k, T_k, Z_k, A_1^k, \dots, A_p^k\}$$

as well as the set

$$\{X_1, \dots, X_d, Y_1, \dots, Y_d, T, Z, A_1, \dots, A_p\}.$$

This gives the following Lie brackets:

$$\begin{aligned} [X_i^k, Y_j^k] &= \delta_{i,j} c_j^k Z_k \text{ mod } \mathfrak{a}_k, \\ [X_i^k, X_j^k] &= 0 \text{ mod } \mathfrak{a}_k \quad \text{and} \\ [Y_i^k, Y_j^k] &= 0 \text{ mod } \mathfrak{a}_k. \end{aligned}$$

The vectors  $Z_k$  and  $T_k$  are central modulo  $\mathfrak{a}_k$ .

Before starting the analysis of  $(\pi_k)_{k \in \mathbb{N}}$ , some notations have to be introduced:

### 4.3 Definitions

Choose for  $j \in \{1, \dots, d\}$  the Schwartz functions  $\eta_j \in \mathcal{S}(\mathbb{R})$  such that  $\|\eta_j\|_{L^2(\mathbb{R})} = 1$  and  $\|\eta_j\|_{L^\infty(\mathbb{R})} \leq 1$ .

Furthermore, for  $x_1, \dots, x_d, y_1, \dots, y_d, t, z, a_1, \dots, a_p \in \mathbb{R}$ , write

$$\begin{aligned} (x)_k &:= (x_1, \dots, x_d)_k := \sum_{j=1}^d x_j X_j^k, \quad (y)_k := (y_1, \dots, y_d)_k := \sum_{j=1}^d y_j Y_j^k, \quad (t)_k := t T_k, \\ (z)_k &:= z Z_k, \quad (\dot{a})_k := (a_1, \dots, a_{\tilde{p}})_k := \sum_{j=1}^{\tilde{p}} a_j A_j^k, \quad (\ddot{a})_k := (a_{\tilde{p}+1}, \dots, a_p)_k := \sum_{j=\tilde{p}+1}^p a_j A_j^k \\ \text{and } (a)_k &:= (\dot{a}, \ddot{a})_k = (a_1, \dots, a_p)_k = \sum_{j=1}^p a_j A_j^k, \end{aligned}$$

whereat  $(\cdot, \dots, \cdot)_k$  is defined to be the  $d$ -,  $\tilde{p}$ -,  $(p - \tilde{p})$ - or the  $p$ -tuple with respect to the bases  $\{X_1^k, \dots, X_d^k\}$ ,  $\{Y_1^k, \dots, Y_d^k\}$ ,  $\{A_1^k, \dots, A_{\tilde{p}}^k\}$ ,  $\{A_{\tilde{p}+1}^k, \dots, A_p^k\}$  and  $\{A_1^k, \dots, A_p^k\}$ , respectively, and let

$$\begin{aligned} (g)_k &:= (x_1, \dots, x_d, y_1, \dots, y_d, t, z, a_1, \dots, a_p)_k := ((x)_k, (y)_k, (t)_k, (z)_k, (\dot{a})_k, (\ddot{a})_k) \\ &= ((x)_k, (h)_k) \\ &= \sum_{j=1}^d x_j X_j^k + \sum_{j=1}^d y_j Y_j^k + t T_k + z Z_k + \sum_{j=1}^p a_j A_j^k, \end{aligned}$$

where  $(h)_k$  is in the polarization  $\mathfrak{p}_k$  and the  $(2d+2+p)$ -tuple  $(\cdot, \dots, \cdot)_k$  is regarded with respect to the basis  $\{X_1^k, \dots, X_d^k, Y_1^k, \dots, Y_d^k, T_k, Z_k, A_1^k, \dots, A_p^k\}$ .

Moreover, define the limits

$$\begin{aligned} (x)_\infty &:= (x_1, \dots, x_d)_\infty := \lim_{k \rightarrow \infty} (x)_k = \sum_{j=1}^d x_j X_j, \quad (y)_\infty := (y_1, \dots, y_d)_\infty := \lim_{k \rightarrow \infty} (y)_k = \sum_{j=1}^d y_j Y_j, \\ (t)_\infty &:= \lim_{k \rightarrow \infty} (t)_k = tT, \quad (z)_\infty := \lim_{k \rightarrow \infty} (z)_k = zZ, \quad (\dot{a})_\infty := (a_1, \dots, a_{\bar{p}})_\infty := \lim_{k \rightarrow \infty} (\dot{a})_k = \sum_{j=1}^{\bar{p}} a_j A_j, \\ (\ddot{a})_\infty &:= (a_{\bar{p}+1}, \dots, a_p)_\infty := \lim_{k \rightarrow \infty} (\ddot{a})_k = \sum_{j=\bar{p}+1}^p a_j A_j, \\ (a)_\infty &:= (\dot{a}, \ddot{a})_\infty = (a_1, \dots, a_p)_\infty := \lim_{k \rightarrow \infty} (a)_k = \sum_{j=1}^p a_j A_j \quad \text{and} \\ (g)_\infty &:= (x, y, t, z, \dot{a}, \ddot{a})_\infty := \lim_{k \rightarrow \infty} (g)_k = \sum_{j=1}^d x_j X_j + \sum_{j=1}^d y_j Y_j + tT + zZ + \sum_{j=1}^p a_j A_j. \end{aligned}$$

Now, the representations  $(\pi_k)_{k \in \mathbb{N}}$  can be computed:

#### 4.4 Formula for $\pi_k$

Let  $f \in L^1(G)$ .

With  $\rho_k := \langle \ell_k, T_k \rangle$ ,  $c^k := (c_1^k, \dots, c_d^k)$  and  $(s)_k := (s_1, \dots, s_d)_k = \sum_{j=1}^d s_j X_j^k$  for  $s_1, \dots, s_d \in \mathbb{R}$ , where again  $(\cdot, \dots, \cdot)_k$  is the  $d$ -tuple with respect to the basis  $\{X_1^k, \dots, X_d^k\}$ , as in (2), the representation  $\pi_k$  acts on  $L^2(G/P_k, \chi_{\ell_k})$  in the following way:

$$\begin{aligned} \pi_k((g)_k) \xi((s)_k) &= \xi((g)_k^{-1} \cdot (s)_k) \\ &= e^{2\pi i \langle \ell_k, -(y)_k - (t)_k - (z)_k - (\dot{a})_k - (\ddot{a})_k + [(s)_k - \frac{1}{2}(x)_k, (y)_k] - \frac{1}{2}[(x)_k, (s)_k] \rangle} \xi((s - x)_k) \\ &= e^{2\pi i \left( -t\rho_k - z\lambda_k + \sum_{j=1}^d \lambda_k c_j^k (s_j - \frac{1}{2}x_j) y_j \right)} \xi((s - x)_k) \\ &= e^{2\pi i (-t\rho_k - z\lambda_k + \lambda_k c^k((s)_k - \frac{1}{2}(x)_k)(y)_k)} \xi((s - x)_k), \end{aligned} \tag{5}$$

since  $\ell_k(Y_j^k) = 0$  for all  $j \in \{1, \dots, d\}$ .

Now, identify  $G$  with  $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\bar{p}} \times \mathbb{R}^{p-\bar{p}} \cong \mathbb{R}^{2d+2+p}$ , let  $\xi \in L^2(\mathbb{R}^d)$  and  $s \in \mathbb{R}^d$ . Moreover, identify  $\pi_k$  with a representation acting on  $L^2(\mathbb{R}^d)$  which will also be called  $\pi_k$ . To stress the dependence on  $k$  of the above fixed function  $f \in L^1(G)$ , denote by  $f_k \in L^1(\mathbb{R}^{2d+2+p})$  the function  $f$  applied to an element in the  $k$ -basis:

$$f_k(g) := f((g)_k).$$

Then,

$$\begin{aligned}
& \pi_k(f)\xi(s) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\bar{p}} \times \mathbb{R}^{p-\bar{p}}} f_k(g) \pi_k(g) \xi(s) dg \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\bar{p}} \times \mathbb{R}^{p-\bar{p}}} f_k(x, y, t, z, \dot{a}, \ddot{a}) e^{2\pi i(-t\rho_k - z\lambda_k + \lambda_k c^k(s - \frac{1}{2}x)y)} \xi(s-x) d(x, y, t, z, \dot{a}, \ddot{a}) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\bar{p}} \times \mathbb{R}^{p-\bar{p}}} f_k(s-x, y, t, z, \dot{a}, \ddot{a}) e^{2\pi i(-t\rho_k - z\lambda_k + \frac{1}{2}\lambda_k c^k(s+x)y)} \xi(x) d(x, y, t, z, \dot{a}, \ddot{a}) \\
&= \int_{\mathbb{R}^d} \hat{f}_k^{2,3,4,5,6} \left( s-x, -\frac{\lambda_k c^k}{2}(s+x), \rho_k, \lambda_k, 0, 0 \right) \xi(x) dx, \tag{6}
\end{aligned}$$

where  $\hat{f}_k^{2,3,4,5,6}$  denotes the Fourier transform in the 2nd, 3rd, 4th, 5th and 6th variable.

## 4.5 First case

First consider the case that  $L((O_k)_k)$  consists of one single limit point  $O$ .

In this case, for every  $k$ ,

$$2d = \dim(O_k) = \dim(O).$$

Thus, the regarded situation occurs if and only if  $\lambda \neq 0$  and  $c_j \neq 0$  for every  $j \in \{1, \dots, d\}$ .

Consider again the above chosen sequence  $(\ell_k)_k$  which converges to  $\ell \in O$ . As the dimensions of the orbits  $O_k$  and  $O$  are the same, there exists a subsequence of  $(\ell_k)_k$  (which will also be denoted by  $(\ell_k)_k$  for simplicity) such that  $\mathfrak{p} := \lim_{k \rightarrow \infty} \mathfrak{p}_{\ell_k}^V$  is a polarization for  $\ell$ , but not necessarily the Vergne polarization. Moreover, define  $P := \exp(\mathfrak{p}) = \lim_{k \rightarrow \infty} P_{\ell_k}^V$  and let

$$\pi := \text{ind}_P^G \chi_\ell.$$

Now, if one identifies the Hilbert spaces  $\mathcal{H}_{\pi_{\ell_k}^V}$  and  $\mathcal{H}_\pi$  of  $\pi_{\ell_k}^V = \text{ind}_{P_{\ell_k}^V}^G \chi_{\ell_k}$  and  $\pi$  with  $L^2(\mathbb{R}^d)$ , from [11], Theorem 2.3, one can conclude that

$$\|\pi_{\ell_k}^V(a) - \pi(a)\|_{op} = \|\text{ind}_{P_{\ell_k}^V}^G \chi_{\ell_k}(a) - \text{ind}_P^G \chi_\ell(a)\|_{op} \xrightarrow{k \rightarrow \infty} 0 \quad \forall a \in C^*(G).$$

Since  $\pi$  and  $\pi_\ell^V = \text{ind}_{P_\ell^V}^G \chi_\ell$  are both induced representations of polarizations and of the same character  $\chi_\ell$ , they are equivalent and hence, there exists a unitary intertwining operator

$$F : \mathcal{H}_{\pi_\ell^V} \cong L^2(\mathbb{R}^d) \rightarrow \mathcal{H}_\pi \cong L^2(\mathbb{R}^d) \text{ such that } F \circ \pi_\ell^V(a) = \pi(a) \circ F \quad \forall a \in C^*(G).$$

Moreover, the two representations  $\pi_k^V = \pi_{\ell_{O_k}}^V = \text{ind}_{P_{\ell_{O_k}}^V}^G \chi_{\ell_{O_k}}$  and  $\pi_{\ell_k}^V = \text{ind}_{P_{\ell_k}^V}^G \chi_{\ell_k}$  are equivalent for every  $k \in \mathbb{N}$  because  $\ell_{O_k}$  and  $\ell_k$  are located in the same coadjoint orbit  $O_k$  and  $\mathfrak{p}_{\ell_{O_k}}^V$  and  $\mathfrak{p}_{\ell_k}^V$  are polarizations. Thus there exist further unitary intertwining operators

$$F_k : \mathcal{H}_{\pi_k^V} \cong L^2(\mathbb{R}^d) \rightarrow \mathcal{H}_{\pi_{\ell_k}^V} \cong L^2(\mathbb{R}^d) \text{ with } F_k \circ \pi_k^V(a) = \pi_{\ell_k}^V(a) \circ F_k \quad \forall a \in C^*(G).$$

Now, define the required operators  $\tilde{\nu}_k$  as

$$\tilde{\nu}_k(\varphi) := F_k^* \circ F \circ \varphi(\pi_\ell^V) \circ F^* \circ F_k \quad \forall \varphi \in CB(S_{i-1}),$$

which makes sense since  $\pi_\ell^V$  is a limit point of the sequence  $(\pi_k^V)_k$  and hence contained in  $S_{i-1}$ , as seen in Chapter 4.1.

As  $\varphi(\pi_\ell^V) \in \mathcal{B}(L^2(\mathbb{R}^d))$  and  $F$  and  $F_k$  are intertwining operators and thus bounded, the image of  $\tilde{\nu}_k$  is contained in  $\mathcal{B}(L^2(\mathbb{R}^d))$ , as requested.

Next, it needs to be shown that  $\tilde{\nu}_k$  is bounded: By the definition of  $\|\cdot\|_{S_{i-1}}$ , one has for every  $\varphi \in CB(S_{i-1})$

$$\|\tilde{\nu}_k(\varphi)\|_{op} = \|F_k^* \circ F \circ \varphi(\pi_\ell^V) \circ F^* \circ F_k\|_{op} \leq \|\varphi(\pi_\ell^V)\|_{op} \leq \|\varphi\|_{S_{i-1}}.$$

In addition, one can easily observe that  $\tilde{\nu}_k$  is involutive: For every  $\varphi \in CB(S_{i-1})$

$$\tilde{\nu}_k(\varphi)^* = (F_k^* \circ F \circ \varphi(\pi_\ell^V) \circ F^* \circ F_k)^* = F_k^* \circ F \circ \varphi^*(\pi_\ell^V) \circ F^* \circ F_k = \tilde{\nu}_k(\varphi^*).$$

Now, the last thing to check is the required convergence of Condition 3(b): For every  $a \in C^*(G)$

$$\begin{aligned} \|\pi_k^V(a) - \tilde{\nu}_k(\mathcal{F}(a)|_{S_{i-1}})\|_{op} &= \|\pi_k^V(a) - F_k^* \circ F \circ \mathcal{F}(a)|_{S_{i-1}}(\pi_\ell^V) \circ F^* \circ F_k\|_{op} \\ &= \|\pi_k^V(a) - F_k^* \circ F \circ \pi_\ell^V(a) \circ F^* \circ F_k\|_{op} \\ &= \|F_k^* \circ \pi_\ell^V(a) \circ F_k - F_k^* \circ \pi(a) \circ F_k\|_{op} \\ &= \|F_k^* \circ (\pi_\ell^V - \pi)(a) \circ F_k\|_{op} \\ &\leq \|\pi_\ell^V(a) - \pi(a)\|_{op} \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Therefore, the representations  $(\pi_k^V)_k$  and the constructed  $(\tilde{\nu}_k)_k$  fulfill Condition 3(b) and thus, in this case, the claim is shown.

## 4.6 Second case

In the second case the situation that  $\lambda = 0$  or  $c_j = 0$  for every  $j \in \{1, \dots, d\}$  must be considered. In this case,

$$\langle \ell_k, [X_j^k, Y_j^k] \rangle = c_j^k \lambda_k \xrightarrow{k \rightarrow \infty} c_j \lambda = 0 \quad \forall j \in \{1, \dots, d\},$$

while  $c_j^k \lambda_k \neq 0$  for every  $k$  and every  $j \in \{1, \dots, d\}$ .

Then  $\ell_{[\mathfrak{g}, \mathfrak{g}]} = 0$  and so every limit orbit  $O$  in the set  $L((O_k)_k)$  has the dimension 0.

As in Calculation (6) in Chapter 4.4, identify  $G$  again with  $\mathbb{R}^{2d+2+p}$ . From now on, this identification will be used most of the time. Only in some cases where one applies  $\ell_k$  or  $\ell$  and thus it is important to know whether one is using the basis depending on  $k$  or the limit basis, the calculation will be done in the above defined bases  $(\cdot)_k$  or  $(\cdot)_\infty$ .

Now, adapt the methods developed in [7] to this given situation.

Let  $s = (s_1, \dots, s_d)$ ,  $\alpha = (\alpha_1, \dots, \alpha_d)$ ,  $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$  and define

$$\eta_{k, \alpha, \beta}(s) = \eta_{k, \alpha, \beta}(s_1, \dots, s_d) := e^{2\pi i \alpha s} \prod_{j=1}^d |\lambda_k c_j^k|^{\frac{1}{4}} \eta_j \left( |\lambda_k c_j^k|^{\frac{1}{2}} \left( s_j + \frac{\beta_j}{\lambda_k c_j^k} \right) \right).$$

Moreover, let  $c_{\alpha, \beta}^k$  be the coefficient function defined by

$$c_{\alpha, \beta}^k(g) := \langle \pi_k(g) \eta_{k, \alpha, \beta}, \eta_{k, \alpha, \beta} \rangle \quad \forall g \in G \cong \mathbb{R}^{2d+2+p}$$

and  $\ell_{\alpha, \beta}$  the linear functional

$$\ell_{\alpha, \beta}(g) = \ell_{\alpha, \beta}(x, y, t, z, a) := \alpha x + \beta y \quad \forall g = (x, y, t, z, a) \in G \cong \mathbb{R}^{2d+2+p}.$$

Then, as in [7], one can show by similar computations that the functions  $c_{\alpha, \beta}^k$  converge uniformly on compacta to the character  $\chi_{\ell + \ell_{\alpha, \beta}}$ .

#### 4.6.1 Definition of the $\nu_k$ 's

For  $0 \neq \tilde{\eta} \in L^2(G/P_k, \chi_{\ell_k}) \cong L^2(\mathbb{R}^d)$  let

$$P_{\tilde{\eta}} : L^2(\mathbb{R}^d) \rightarrow \mathbb{C}\tilde{\eta}, \quad \xi \mapsto \tilde{\eta}\langle \xi, \tilde{\eta} \rangle.$$

Then  $P_{\tilde{\eta}}$  is the orthogonal projection onto the space  $\mathbb{C}\tilde{\eta}$ .

Let  $h \in C^*(G/U, \chi_{\ell})$ . Again, identify  $G/U$  with  $\mathbb{R}^d \times \mathbb{R}^d \cong \mathbb{R}^{2d}$  and as already introduced in Chapter 4.4 in order to show the dependence on  $k$ , here the utilization of the limit basis will be expressed by an index  $\infty$  if necessary:

$$h_{\infty}(x, y) := h((x, y)_{\infty}).$$

Now,  $\hat{h}_{\infty}$  can be seen as a function in  $C_0(\ell + \mathfrak{u}^{\perp}) \cong C_0(\mathbb{R}^{2d})$  and, using this identification, define the linear operator

$$\nu_k(h) := \int_{\mathbb{R}^{2d}} \hat{h}_{\infty}(\tilde{x}, \tilde{y}) P_{\eta_{k, \tilde{x}, \tilde{y}}} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d |\lambda_k c_j^k|}.$$

Then, the following proposition holds:

**Proposition 4.1.**

1. For every  $k \in \mathbb{N}$  and  $h \in \mathcal{S}(G/U, \chi_{\ell})$  the integral defining  $\nu_k(h)$  converges in the operator norm.
2. The operator  $\nu_k(h)$  is compact and  $\|\nu_k(h)\|_{op} \leq \|h\|_{C^*(G/U, \chi_{\ell})}$ .
3.  $\nu_k$  is involutive, i.e.  $\nu_k(h)^* = \nu_k(h^*)$  for every  $h \in C^*(G/U, \chi_{\ell})$ .

Proof:

1. Let  $h \in \mathcal{S}(G/U, \chi_{\ell}) \cong \mathcal{S}(\mathbb{R}^{2d})$ . Since

$$\|P_{\eta_{k, \alpha, \beta}}\|_{op} = \|\eta_{k, \alpha, \beta}\|_2^2 = 1,$$

one can estimate the operator norm of  $\nu_k(h)$  as follows:

$$\begin{aligned} \|\nu_k(h)\|_{op} &= \left\| \int_{\mathbb{R}^{2d}} \hat{h}_{\infty}(\tilde{x}, \tilde{y}) P_{\eta_{k, \tilde{x}, \tilde{y}}} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d |\lambda_k c_j^k|} \right\|_{op} \\ &\leq \int_{\mathbb{R}^{2d}} |\hat{h}_{\infty}(\tilde{x}, \tilde{y})| \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d |\lambda_k c_j^k|} = \frac{\|\hat{h}\|_{L^1(\mathbb{R}^{2d})}}{\prod_{j=1}^d |\lambda_k c_j^k|}. \end{aligned}$$

Therefore, the convergence of the integral  $\nu_k(h)$  in the operator norm is shown for  $h \in \mathcal{S}(\mathbb{R}^{2d}) \cong \mathcal{S}(G/U, \chi_{\ell})$ .

2. First, let  $h \in \mathcal{S}(G/U, \chi_{\ell}) \cong \mathcal{S}(\mathbb{R}^{2d})$ . Define for  $s = (s_1, \dots, s_d) \in \mathbb{R}^d$

$$\eta_{k, \beta}(s) := \prod_{j=1}^d |\lambda_k c_j^k|^{\frac{1}{4}} \eta_j \left( |\lambda_k c_j^k|^{\frac{1}{2}} \left( s_j + \frac{\beta_j}{\lambda_k c_j^k} \right) \right).$$

Then

$$\eta_{k, \alpha, \beta}(s) = e^{2\pi i \alpha s} \eta_{k, \beta}(s)$$

and thus one has for  $\xi \in \mathcal{S}(\mathbb{R}^d)$  and  $s \in \mathbb{R}^d$

$$\begin{aligned}
& \nu_k(h)\xi(s) \\
&= \int_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x}, \tilde{y}) \langle \xi, \eta_{k, \tilde{x}, \tilde{y}} \rangle \eta_{k, \tilde{x}, \tilde{y}}(s) \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d |\lambda_k c_j^k|} \\
&= \int_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x}, \tilde{y}) \left( \int_{\mathbb{R}^d} \xi(r) \overline{\eta}_{k, \tilde{x}, \tilde{y}}(r) dr \right) \eta_{k, \tilde{x}, \tilde{y}}(s) \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d |\lambda_k c_j^k|} \\
&= \int_{\mathbb{R}^{2d}} \hat{h}_\infty(\tilde{x}, \tilde{y}) \left( \int_{\mathbb{R}^d} \xi(r) e^{-2\pi i \tilde{x} r} \overline{\eta}_{k, \tilde{y}}(r) dr \right) e^{2\pi i \tilde{x} s} \eta_{k, \tilde{y}}(s) \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=1}^d |\lambda_k c_j^k|} \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{h}_\infty(\tilde{x}, \tilde{y}) e^{2\pi i \tilde{x}(s-r)} \xi(r) \overline{\eta}_{k, \tilde{y}}(r) dr \eta_{k, \tilde{y}}(s) \frac{d\tilde{x} d\tilde{y}}{\prod_{j=1}^d |\lambda_k c_j^k|} \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{h}_\infty^2(s-r, \tilde{y}) \xi(r) \overline{\eta}_{k, \tilde{y}}(r) \eta_{k, \tilde{y}}(s) \frac{d\tilde{y}}{\prod_{j=1}^d |\lambda_k c_j^k|} dr. \tag{7}
\end{aligned}$$

Hence, as the kernel function

$$h_K(s, r) := \int_{\mathbb{R}^d} \hat{h}_\infty^2(s-r, \tilde{y}) \overline{\eta}_{k, \tilde{y}}(r) \eta_{k, \tilde{y}}(s) \frac{d\tilde{y}}{\prod_{j=1}^d |\lambda_k c_j^k|}$$

of  $\nu_k(h)$  is in  $\mathcal{S}(\mathbb{R}^{2d})$ ,  $\nu_k(h)$  is a compact operator.

Now it will be shown that

$$\|\nu_k(h)\|_{op} \leq \|\hat{h}\|_\infty :$$

For  $\xi \in \mathcal{S}(\mathbb{R}^d)$  one has similar as in [7]

$$\begin{aligned}
& \|\nu_k(h)\xi\|_2^2 \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{h}_\infty^2(s-r, \tilde{y}) \xi(r) \overline{\eta}_{k, \tilde{y}}(r) \eta_{k, \tilde{y}}(s) \frac{d\tilde{y}}{\prod_{j=1}^d |\lambda_k c_j^k|} dr \right|^2 ds \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \hat{h}_\infty^2(\cdot, \tilde{y}) * (\xi \overline{\eta}_{k, \tilde{y}})(s) \eta_{k, \tilde{y}}(s) \frac{d\tilde{y}}{\prod_{j=1}^d |\lambda_k c_j^k|} \right|^2 ds \\
&\stackrel{\text{Cauchy-Schwarz}}{\leq} \prod_{j=1}^d |\lambda_k c_j^k| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\hat{h}_\infty^2(\cdot, \tilde{y}) * (\xi \overline{\eta}_{k, \tilde{y}})(s)|^2 ds \frac{d\tilde{y}}{\prod_{j=1}^d |\lambda_k c_j^k|^2} \\
&\stackrel{\text{Plancherel}}{\leq} \|\hat{h}\|_\infty^2 \prod_{j=1}^d \frac{1}{|\lambda_k c_j^k|} \int_{\mathbb{R}^d} \|\xi \overline{\eta}_{k, \tilde{y}}\|_2^2 d\tilde{y} \\
&\stackrel{\|\eta_j\|_2=1}{=} \|\hat{h}\|_\infty^2 \|\xi\|_2^2.
\end{aligned}$$



Thus, since  $\mathcal{S}(\mathbb{R}^d)$  is dense in  $L^2(\mathbb{R}^d)$ ,

$$\|\nu_k(h)\|_{op} = \sup_{\substack{\xi \in L^2(\mathbb{R}^d), \\ \|\xi\|_2=1}} \|\nu_k(h)(\xi)\|_2 \leq \|\hat{h}\|_\infty$$

for  $h \in \mathcal{S}(\mathbb{R}^{2d}) \cong \mathcal{S}(G/U, \chi_\ell)$ . Therefore, with the density of  $\mathcal{S}(G/U, \chi_\ell)$  in  $C^*(G/U, \chi_\ell)$ , one gets the compactness of the operator  $\nu_k(h)$  for  $h \in C^*(G/U, \chi_\ell)$ , as well as the desired inequality

$$\|\nu_k(h)\|_{op} \leq \|\hat{h}\|_\infty = \|h\|_{C^*(G/U, \chi_\ell)}.$$

3. The proof of the involutivity of  $\nu_k$  is straightforward. □

This proposition firstly shows that the image of the operator  $\nu_k$  is located in  $\mathcal{B}(L^2(\mathbb{R}^d)) = \mathcal{B}(\mathcal{H}_i)$  as required in Condition 3(b). Secondly, the proposition gives the boundedness and the involutivity of the linear mappings  $\nu_k$  for every  $k \in \mathbb{N}$ . For the analysis of the sequence  $(\pi_k)_k$ , it remains to show the convergence condition.

#### 4.6.2 Theorem - Second Case

##### Theorem 4.2.

Define as in Subsection 2.4

$$\begin{aligned} p_{G/U} &: L^1(G) \rightarrow L^1(G/U, \chi_\ell), \\ p_{G/U}(f)(\tilde{g}) &:= \int_U f(\tilde{g}u) \chi_\ell(u) du \quad \forall \tilde{g} \in G \quad \forall f \in L^1(G) \end{aligned}$$

and canonically extend  $p_{G/U}$  to a mapping going from  $C^*(G)$  to  $C^*(G/U, \chi_\ell)$ . Furthermore let  $a \in C^*(G)$ . Then

$$\lim_{k \rightarrow \infty} \|\pi_k(a) - \nu_k(p_{G/U}(a))\|_{op} = 0.$$

Proof:

For  $u = (t, z, \dot{a}, \ddot{a})_\infty \in U = \text{span}\{T, Z, A_1, \dots, A_{\bar{p}}, A_{\bar{p}+1}, \dots, A_p\}$

$$\chi_\ell(u) = e^{-2\pi i \langle \ell, (t, z, \dot{a}, \ddot{a})_\infty \rangle} = e^{-2\pi i (t\rho + z\lambda)}$$

and therefore, identifying  $U$  again with  $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\bar{p}} \times \mathbb{R}^{p-\bar{p}}$  and  $L^1(G/U, \chi_\ell)$  with  $L^1(\mathbb{R}^{2d})$ , for  $f \in L^1(G) \cong L^1(\mathbb{R}^{2d+2+p})$  and  $\tilde{g} = (\tilde{x}, \tilde{y}, 0, 0, 0, 0) \in \mathbb{R}^{2d}$  one has

$$\begin{aligned} (p_{G/U}(f))_\infty(\tilde{g}) &= \int_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\bar{p}} \times \mathbb{R}^{p-\bar{p}}} f_\infty(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{z}, \tilde{\dot{a}}, \tilde{\ddot{a}}) e^{-2\pi i (\tilde{t}\rho + \tilde{z}\lambda)} d(0, 0, \tilde{t}, \tilde{z}, \tilde{\dot{a}}, \tilde{\ddot{a}}) \\ &= \hat{f}_\infty^{3,4,5,6}(\tilde{x}, \tilde{y}, \rho, \lambda, 0, 0), \end{aligned} \tag{8}$$

whereat  $f_\infty(\tilde{x}, \tilde{y}, \rho, \lambda, 0, 0) = f((\tilde{x}, \tilde{y}, \rho, \lambda, 0, 0)_\infty)$ .

Now, let  $f \in \mathcal{S}(G) \cong \mathcal{S}(\mathbb{R}^{2d+2+p})$  such that its Fourier transform in  $[\mathfrak{g}, \mathfrak{g}]$  has a compact support on  $G \cong \mathbb{R}^{2d+2+p}$ . If one then writes the elements  $g$  of  $G$  as  $g = (x, y, t, z, \dot{a}, \ddot{a})$  like above, whereat

$$\begin{aligned} x &\in \text{span}\{X_1\} \times \dots \times \text{span}\{X_d\}, \quad y \in \text{span}\{Y_1\} \times \dots \times \text{span}\{Y_d\}, \quad t \in \text{span}\{T\}, \\ z &\in \text{span}\{Z\}, \quad \dot{a} \in \text{span}\{A_1\} \times \dots \times \text{span}\{A_{\bar{p}}\}, \quad \ddot{a} \in \text{span}\{A_{\bar{p}+1}\} \times \dots \times \text{span}\{A_p\} \end{aligned}$$

or, respectively

$$x \in \text{span}\{X_1^k\} \times \dots \times \text{span}\{X_d^k\}, \quad y \in \text{span}\{Y_1^k\} \times \dots \times \text{span}\{Y_d^k\}, \quad t \in \text{span}\{T_k\},$$

$$z \in \text{span}\{Z_k\}, \quad \dot{a} \in \text{span}\{A_1^k\} \times \dots \times \text{span}\{A_{\bar{p}}^k\}, \quad \ddot{a} \in \text{span}\{A_{\bar{p}+1}^k\} \times \dots \times \text{span}\{A_p^k\},$$

this means that the partial Fourier transform  $\hat{f}^{4,5}$  has a compact support in  $G$ , since  $[\mathfrak{g}, \mathfrak{g}] = \text{span}\{Z_k, A_1^k, \dots, A_{\bar{p}}^k\} = \text{span}\{Z, A_1, \dots, A_{\bar{p}}\}$ .

Moreover, let  $s \in \mathbb{R}^d$  and define

$$\eta_{k,0}(s) := \prod_{j=1}^d |\lambda_k c_j^k|^{\frac{1}{4}} \eta_j \left( |\lambda_k c_j^k|^{\frac{1}{2}}(s_j) \right).$$

(Compare the definition of  $\eta_{k,\beta}$  in the last proof.)

If  $\xi \in \mathcal{S}(\mathbb{R}^d)$ , one has

$$\begin{aligned} & (\pi_k(f) - \nu_k(p_{G/U}(f)))\xi(s) \\ & \stackrel{(6),(7)}{=} \int_{\mathbb{R}^d} \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho_k, \lambda_k, 0, 0 \right) \xi(r) \, dr \\ & - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widehat{p_{G/U}(f)}_{\infty}^2(s - r, \tilde{y}) \xi(r) \overline{\eta}_{k,\tilde{y}}(r) \eta_{k,\tilde{y}}(s) \frac{d\tilde{y}}{\prod_{j=1}^d |\lambda_k c_j^k|} \, dr \\ & \stackrel{\|\eta_{k,0}\|_2=1}{\stackrel{(8)}}{=} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho_k, \lambda_k, 0, 0 \right) \xi(r) \overline{\eta}_{k,0}(\tilde{y}) \eta_{k,0}(\tilde{y}) \, d\tilde{y} dr \\ & - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{f}_{\infty}^{2,3,4,5,6}(s - r, \tilde{y}, \rho, \lambda, 0, 0) \xi(r) \overline{\eta}_{k,\tilde{y}}(r) \eta_{k,\tilde{y}}(s) \frac{d\tilde{y}}{\prod_{j=1}^d |\lambda_k c_j^k|} \, dr \\ & = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho_k, \lambda_k, 0, 0 \right) \xi(r) \overline{\eta}_{k,0}(\tilde{y}) \eta_{k,0}(\tilde{y}) \, d\tilde{y} dr \\ & - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{f}_{\infty}^{2,3,4,5,6}(s - r, \lambda_k c^k(\tilde{y} - s), \rho, \lambda, 0, 0) \xi(r) \overline{\eta}_{k,0}(\tilde{y} + r - s) \eta_{k,0}(\tilde{y}) \, d\tilde{y} dr. \end{aligned}$$

The just obtained integrals are now divided into five parts. To do so, new functions  $q_k$ ,  $u_k$ ,  $v_k$ ,  $n_k$  and  $w_k$  are defined:

$$\begin{aligned} q_k(s, \tilde{y}) &:= \int_{\mathbb{R}^d} \xi(r) \overline{\eta}_{k,0}(\tilde{y} + r - s) \left( \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \textcolor{red}{\rho}_k, \lambda_k, 0, 0 \right) \right. \\ &\quad \left. - \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \textcolor{red}{\rho}, \lambda_k, 0, 0 \right) \right) dr, \\ u_k(s, \tilde{y}) &:= \int_{\mathbb{R}^d} \xi(r) \overline{\eta}_{k,0}(\tilde{y} + r - s) \left( \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho, \textcolor{red}{\lambda}_k, 0, 0 \right) \right. \\ &\quad \left. - \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho, \textcolor{red}{\lambda}, 0, 0 \right) \right) dr, \end{aligned}$$

$$v_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(r) \bar{\eta}_{k,0}(\tilde{y} + r - s) \left( \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho, \lambda, 0, 0 \right) - \hat{f}_k^{2,3,4,5,6} \left( s - r, \lambda_k c^k(\tilde{y} - s), \rho, \lambda, 0, 0 \right) \right) dr,$$

$$n_k(s, \tilde{y}) := \int_{\mathbb{R}^d} \xi(r) \bar{\eta}_{k,0}(\tilde{y} + r - s) \left( \hat{f}_k^{2,3,4,5,6} \left( s - r, \lambda_k c^k(\tilde{y} - s), \rho, \lambda, 0, 0 \right) - \hat{f}_\infty^{2,3,4,5,6} \left( s - r, \lambda_k c^k(\tilde{y} - s), \rho, \lambda, 0, 0 \right) \right) dr$$

and

$$w_k(s) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi(r) \eta_{k,0}(\tilde{y}) (\bar{\eta}_{k,0}(\tilde{y}) - \bar{\eta}_{k,0}(\tilde{y} + r - s)) \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho_k, \lambda_k, 0, 0 \right) dr d\tilde{y}.$$

Then,

$$\begin{aligned} (\pi_k(f) - \nu_k(p_{G/U}(f)))\xi(s) &= \int_{\mathbb{R}^d} q_k(s, \tilde{y}) \eta_{k,0}(\tilde{y}) d\tilde{y} + \int_{\mathbb{R}^d} u_k(s, \tilde{y}) \eta_{k,0}(\tilde{y}) d\tilde{y} \\ &+ \int_{\mathbb{R}^d} v_k(s, \tilde{y}) \eta_{k,0}(\tilde{y}) d\tilde{y} + \int_{\mathbb{R}^d} n_k(s, \tilde{y}) \eta_{k,0}(\tilde{y}) d\tilde{y} + w_k(s). \end{aligned}$$

In order to show that

$$\|\pi_k(f) - \nu_k(p_{G/U}(f))\|_{op} \xrightarrow{k \rightarrow \infty} 0,$$

it suffices to prove that there are  $\kappa_k, \gamma_k, \delta_k, \omega_k$  and  $\epsilon_k$  which are going to 0 for  $k \rightarrow \infty$ , such that

$$\begin{aligned} \|q_k\|_2 &\leq \kappa_k \|\xi\|_2, \quad \|u_k\|_2 \leq \gamma_k \|\xi\|_2, \quad \|v_k\|_2 \leq \delta_k \|\xi\|_2, \quad \|n_k\|_2 \leq \omega_k \|\xi\|_2 \\ \text{and} \quad \|w_k\|_2 &\leq \epsilon_k \|\xi\|_2. \end{aligned}$$

First, regard the last factor of the function  $q_k$ :

$$\begin{aligned} &\hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho_k, \lambda_k, 0, 0 \right) - \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho, \lambda_k, 0, 0 \right) \\ &= (\rho_k - \rho) \int_0^1 \partial_3 \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho + t(\rho_k - \rho), \lambda_k, 0, 0 \right) dt. \end{aligned}$$

Thus, since  $f$  is a Schwartz function, one can find a constant  $C_1 > 0$  (depending on  $f$ ), such that

$$\begin{aligned} &\left| \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho_k, \lambda_k, 0, 0 \right) - \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho, \lambda_k, 0, 0 \right) \right| \\ &\leq |\rho_k - \rho| \frac{C_1}{(1 + \|s - r\|)^{2d}}. \end{aligned}$$

Hence, one gets the following estimation for  $q_k$ :

$$\begin{aligned}
\|q_k\|_2^2 &= \int_{\mathbb{R}^{2d}} |q_k(s, \tilde{y})|^2 d(s, \tilde{y}) \\
&\leq \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} |\xi(r) \overline{\eta}_{k,0}(\tilde{y} + r - s)| |\rho_k - \rho| \frac{C_1}{(1 + \|s - r\|)^{2d}} dr \right)^2 d(s, \tilde{y}) \\
&\stackrel{\text{Cauchy-Schwarz}}{\leq} C_1^2 |\rho_k - \rho|^2 \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} \left| \frac{\xi(r) \overline{\eta}_{k,0}(\tilde{y} + r - s)}{(1 + \|s - r\|)^d} \right|^2 dr \right) \\
&\quad \left( \int_{\mathbb{R}^d} \left( \frac{1}{(1 + \|s - r\|)^d} \right)^2 dr \right) d(s, \tilde{y}) \\
&= C_1' |\rho_k - \rho|^2 \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1 + \|s - r\|)^{2d}} |\eta_{k,0}(\tilde{y} + r - s)|^2 d(r, s, \tilde{y}) \\
&\stackrel{\|\eta_{k,0}\|_2=1}{\leq} C_1'' |\rho_k - \rho|^2 \|\xi\|_2^2,
\end{aligned}$$

whereat  $C_1' > 0$  and  $C_1'' > 0$  are matching constants depending on  $f$ . Thus, for  $\kappa_k := \sqrt{C_1''} |\rho_k - \rho|$ ,  $\kappa_k \xrightarrow{k \rightarrow \infty} 0$ , since  $\rho_k \xrightarrow{k \rightarrow \infty} \rho$ , and

$$\|q_k\|_2 \leq \kappa_k \|\xi\|_2.$$

As  $\lambda_k \xrightarrow{k \rightarrow \infty} \lambda$ , the estimation for the function  $u_k$  can be done analogously.

Now, regard  $v_k$ . Like for  $q_k$  and  $u_k$ , one has

$$\begin{aligned}
&\hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2} (s + r), \rho, \lambda, 0, 0 \right) - \hat{f}_k^{2,3,4,5,6} (s - r, \lambda_k c^k (\tilde{y} - s), \rho, \lambda, 0, 0) \\
&= \lambda_k c^k \left( \frac{1}{2} (r - s) - (r - s + \tilde{y}) \right) \\
&\cdot \int_0^1 \partial_2 \hat{f}_k^{2,3,4,5,6} \left( s - r, \lambda_k c^k (\tilde{y} - s) + t \lambda_k c^k \left( \frac{1}{2} (r - s) - (r - s + \tilde{y}) \right), \rho, \lambda, 0, 0 \right) dt,
\end{aligned}$$

whereat  $\cdot$  is the scalar product, and hence there exists again an on  $f$  depending constant  $C_3$  such that

$$\begin{aligned}
&\left| \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2} (s + r), \rho, \lambda, 0, 0 \right) - \hat{f}_k^{2,3,4,5,6} (s - r, \lambda_k c^k (\tilde{y} - s), \rho, \lambda, 0, 0) \right| \\
&\leq |\lambda_k| \left( \|c^k (r - s)\| + \|c^k (r - s + \tilde{y})\| \right) \frac{C_3}{(1 + \|s - r\|)^{2d+1}}.
\end{aligned}$$

Therefore, defining  $\tilde{\eta}_j(t) := \|t\| \eta_j(t)$ , one gets a similar estimation for  $v_k$ :

$$\begin{aligned}
& \|v_k\|_2^2 \\
&= \int_{\mathbb{R}^{2d}} |v_k(s, \tilde{y})|^2 d(s, \tilde{y}) \\
&\leq \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} |\xi(r) \bar{\eta}_{k,0}(\tilde{y} + r - s)| |\lambda_k| \left( \|c^k(r - s)\| + \|c^k(r - s + \tilde{y})\| \right) \right. \\
&\quad \left. \frac{C_3}{(1 + \|s - r\|)^{2d+1}} dr \right)^2 d(s, \tilde{y}) \\
&\stackrel{\text{Cauchy-Schwarz}}{\leq} C_3' \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1 + \|s - r\|)^{2d+2}} |\eta_{k,0}(\tilde{y} + r - s)|^2 |\lambda_k|^2 \\
&\quad \left( \|c^k(r - s)\| + \|c^k(r - s + \tilde{y})\| \right)^2 d(r, s, \tilde{y}) \\
&\leq 2C_3' \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1 + \|s - r\|)^{2d+2}} |\eta_{k,0}(\tilde{y} + r - s)|^2 |\lambda_k|^2 \|c^k(r - s + \tilde{y})\|^2 d(r, s, \tilde{y}) \\
&+ 2C_3' \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1 + \|s - r\|)^{2d+2}} |\eta_{k,0}(\tilde{y} + r - s)|^2 |\lambda_k|^2 \|c^k(r - s)\|^2 d(r, s, \tilde{y}) \\
&\leq 2C_3' |\lambda_k|^2 \left( \int_{\mathbb{R}^{2d}} \frac{|\xi(r)|^2}{(1 + \|s - r\|)^{2d+2}} d(r, s) \right) \left( \int_{\mathbb{R}^d} \|c^k \tilde{y}\|^2 |\eta_{k,0}(\tilde{y})|^2 d\tilde{y} \right) \\
&+ 2C_3' \|\lambda_k c^k\|^2 \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1 + \|s - r\|)^{2d}} |\eta_{k,0}(\tilde{y} + r - s)|^2 d(r, s, \tilde{y}) \\
&\leq 2C_3' \|\lambda_k c^k\| \left( \int_{\mathbb{R}^{2d}} \frac{|\xi(r)|^2}{(1 + \|s - r\|)^{2d+2}} d(r, s) \right) \\
&\quad \left( \prod_{j=1}^d |\lambda_k c_j^k|^{\frac{1}{2}} \int_{\mathbb{R}} \left\| |\lambda_k c_j^k|^{\frac{1}{2}} \tilde{y}_j \right\|^2 \left| \eta_j(|\lambda_k c_j^k|^{\frac{1}{2}}(\tilde{y}_j)) \right|^2 d\tilde{y}_j \right) \\
&+ 2C_3' \|\lambda_k c^k\|^2 \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1 + \|s - r\|)^{2d}} |\eta_{k,0}(\tilde{y} + r - s)|^2 d(r, s, \tilde{y}) \\
&\stackrel{\|\eta_{k,0}\|_2=1}{\leq} 2C_3' \|\lambda_k c^k\| \|\xi\|_2^2 \left( \prod_{j=1}^d \int_{\mathbb{R}} |\tilde{\eta}_j(\tilde{y}_j)|^2 d\tilde{y}_j \right) + 2C_3' \|\lambda_k c^k\|^2 \|\xi\|_2^2 \\
&= C_3'' \left( \prod_{j=1}^d \|\tilde{\eta}_j\|_2^2 + \|\lambda_k c^k\| \right) \|\lambda_k c^k\| \|\xi\|_2^2
\end{aligned}$$

with constants  $C_3' > 0$  and  $C_3'' > 0$ , again depending on  $f$ .

Now, since  $\lambda_k c^k \xrightarrow{k \rightarrow \infty} 0$ ,  $\delta_k := \left( C_3'' \left( \prod_{j=1}^d \|\tilde{\eta}_j\|_2^2 + \|\lambda_k c^k\| \right) \|\lambda_k c^k\| \right)^{\frac{1}{2}}$  fulfills  $\delta_k \xrightarrow{k \rightarrow \infty} 0$  and

$$\|v_k\|_2 \leq \delta_k \|\xi\|_2.$$

For the estimation of  $n_k$ , the fact that the Fourier transform in  $[\mathbf{g}, \mathbf{g}]$ , i.e. in the 4th and 5th

variable,  $\hat{f}^{4,5} =: \tilde{f}$  has a compact support will be needed. Therefore, let the support of  $f$  be located in the compact set

$$K_1 \times K_2 \times K_3 \times K_4 \times K_5 \times K_6 \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\tilde{p}} \times \mathbb{R}^{p-\tilde{p}}$$

and let  $K := K_2 \times K_3 \times K_6 \subset \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{p-\tilde{p}}$ .

Furthermore, since a Fourier transform is independent of the choice of the basis, the value of  $\tilde{f}$  on  $[\mathfrak{g}, \mathfrak{g}] = \text{span}\{Z_k, A_1^k, \dots, A_{\tilde{p}}^k\} = \text{span}\{Z, A_1, \dots, A_{\tilde{p}}\}$  expressed in the  $k$ -basis and its value expressed in the limit basis are the same:

$$f(\cdot, \cdot, \cdot, (z, \dot{a})_k, \cdot) = f(\cdot, \cdot, \cdot, (z, \dot{a})_\infty, \cdot).$$

So, in the course of this proof, the limit basis will be chosen for the representation of the 4th and 5th position of an element  $g$ . Then

$$\begin{aligned} & \hat{f}_k^{2,3,4,5,6}(s-r, \lambda_k c^k(\tilde{y}-s), \rho, \lambda, 0, 0) - \hat{f}_\infty^{2,3,4,5,6}(s-r, \lambda_k c^k(\tilde{y}-s), \rho, \lambda, 0, 0) \\ &= \int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{p-\tilde{p}}} \left( \tilde{f}_k(s-r, y, t, \lambda, 0, \ddot{a}) - \tilde{f}_\infty(s-r, y, t, \lambda, 0, \ddot{a}) \right) e^{-2\pi i(\lambda_k c^k(\tilde{y}-s)y + \rho t)} d(y, t, \ddot{a}) \\ &= \int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{p-\tilde{p}}} \left( \tilde{f}((s-r, y, t)_k, (\lambda, 0)_\infty, (\ddot{a})_k) - \tilde{f}((s-r, y, t, \lambda, 0, \ddot{a})_\infty) \right) \\ & \quad e^{-2\pi i(\lambda_k c^k(\tilde{y}-s)y + \rho t)} d(y, t, \ddot{a}) \\ &= \int_K \left( \tilde{f}((s-r, y, t)_k, (\lambda, 0)_\infty, (\ddot{a})_k) - \tilde{f}((s-r, y, t, \lambda, 0, \ddot{a})_\infty) \right) e^{-2\pi i(\lambda_k c^k(\tilde{y}-s)y + \rho t)} d(y, t, \ddot{a}). \end{aligned}$$

Furthermore,

$$\begin{aligned} & \tilde{f}((s-r, y, t)_k, (\lambda, 0)_\infty, (\ddot{a})_k) - \tilde{f}((s-r, y, t, \lambda, 0, \ddot{a})_\infty) \\ &= \tilde{f}\left(\sum_{i=1}^d (s_i - r_i) X_i^k + \sum_{i=1}^d y_i Y_i^k + t T_k + \lambda Z + \sum_{i=\tilde{p}+1}^p a_i A_i^k\right) \\ &- \tilde{f}\left(\sum_{i=1}^d (s_i - r_i) X_i + \sum_{i=1}^d y_i Y_i + t T + \lambda Z + \sum_{i=\tilde{p}+1}^p a_i A_i\right) \\ &= \left(\sum_{i=1}^d (s_i - r_i)(X_i^k - X_i) + \sum_{i=1}^d y_i(Y_i^k - Y_i) + t(T_k - T) + \sum_{i=\tilde{p}+1}^p a_i(A_i^k - A_i)\right) \\ &\cdot \int_0^1 \partial \tilde{f}\left(\sum_{i=1}^d (s_i - r_i) X_i + \sum_{i=1}^d y_i Y_i + t T + \lambda Z + \sum_{i=\tilde{p}+1}^p a_i A_i\right. \\ & \quad \left.+ \tilde{t}\left(\sum_{i=1}^d (s_i - r_i)(X_i^k - X_i) + \sum_{i=1}^d y_i(Y_i^k - Y_i) + t(T_k - T) + \sum_{i=\tilde{p}+1}^p a_i(A_i^k - A_i)\right)\right) d\tilde{t}. \end{aligned}$$

Since

$$(X_1^k, \dots, X_d^k, Y_1^k, \dots, Y_d^k, T_k, Z_k, A_1^k, \dots, A_p^k) \xrightarrow{k \rightarrow \infty} (X_1, \dots, X_d, Y_1, \dots, Y_d, T, Z, A_1, \dots, A_p),$$

there exist  $\omega_k^i \xrightarrow{k \rightarrow \infty} 0$  for  $i \in \{1, \dots, 4\}$  and an on  $f$  depending constant  $C_4 > 0$  such that

$$\begin{aligned} & \left| \tilde{f}((s-r, y, t)_k, (\lambda, 0)_\infty, (\ddot{a})_k) - \tilde{f}((s-r, y, t, \lambda, 0, \ddot{a})_\infty) \right| \\ & \leq \left( \|s-r\| \omega_k^1 + \|y\| \omega_k^2 + |t| \omega_k^3 + \|\ddot{a}\| \omega_k^4 \right) \frac{C_4}{(1 + \|s-r\|)^{2d+1}}. \end{aligned}$$

Now, with the help of the two calculations above,  $\|n_k\|_2^2$  can be estimated:

$$\begin{aligned}
& \|n_k\|_2^2 \\
&= \int_{\mathbb{R}^{2d}} |n_k(s, \tilde{y})|^2 d(s, \tilde{y}) \\
&= \int_{\mathbb{R}^{2d}} \left| \int_{\mathbb{R}^d} \xi(r) \bar{\eta}_{k,0}(\tilde{y} + r - s) \left( \hat{f}_k^{2,3,4,5,6}(s-r, \lambda_k c^k(\tilde{y}-s), \rho, \lambda, 0, 0) \right. \right. \\
&\quad \left. \left. - \hat{f}_\infty^{2,3,4,5,6}(s-r, \lambda_k c^k(\tilde{y}-s), \rho, \lambda, 0, 0) \right) dr \right|^2 d(s, \tilde{y}) \\
&\leq \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} |\xi(r) \bar{\eta}_{k,0}(\tilde{y} + r - s)| \int_K \left( \|s-r\| \omega_k^1 + \|y\| \omega_k^2 + |t| \omega_k^3 + \|\ddot{a}\| \omega_k^4 \right) \right. \\
&\quad \left. \frac{C_4}{(1 + \|s-r\|)^{2d+1}} |e^{-2\pi i(\lambda_k c^k(\tilde{y}-s)y + \rho t)}| d(y, t, \ddot{a}) dr \right)^2 d(s, \tilde{y}) \\
&= C_4^2 \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} \frac{1}{(1 + \|s-r\|)^{2d+1}} |\xi(r)| |\bar{\eta}_{k,0}(\tilde{y} + r - s)| \left( \|s-r\| \omega_k^5 + \omega_k^6 \right) dr \right)^2 d(s, \tilde{y}) \\
&\stackrel{\text{Cauchy-Schwarz}}{\leq} C_4^2 \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} \frac{1}{(1 + \|s-r\|)^{2d+2}} |\xi(r)|^2 |\bar{\eta}_{k,0}(\tilde{y} + r - s)|^2 \left( \|s-r\| \omega_k^5 + \omega_k^6 \right)^2 dr \right) \\
&\quad \left( \int_{\mathbb{R}^d} \frac{1}{(1 + \|s-r\|)^{2d}} dr \right) d(s, \tilde{y}) \\
&\leq C_4' \omega_k^7 \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^d} \frac{1}{(1 + \|s-r\|)^{2d}} |\xi(r)|^2 |\bar{\eta}_{k,0}(\tilde{y} + r - s)|^2 dr d(s, \tilde{y}) \\
&\stackrel{\| \eta_{k,0} \|_2 = 1}{=} C_4'' \omega_k^7 \|\xi\|_2^2
\end{aligned}$$

with constants  $C_4' > 0$  and  $C_4'' > 0$  depending on  $f$  and  $\omega_k^i \xrightarrow{k \rightarrow \infty} 0$  for  $i \in \{5, \dots, 7\}$ . Thus,  $\omega_k := \sqrt{C_4'' \omega_k^7}$  fulfills  $\omega_k \xrightarrow{k \rightarrow \infty} 0$  and

$$\|n_k\|_2 \leq \omega_k \|\xi\|_2.$$

Last, it still remains to examine  $w_k$ :

$$\begin{aligned}
& \bar{\eta}_{k,0}(\tilde{y}) - \bar{\eta}_{k,0}(\tilde{y} + r - s) \\
&= \sum_{j=1}^d (r_j - s_j) \int_0^1 \partial_j \bar{\eta}_{k,0}(\tilde{y} + t(r - s)) dt \\
&= \sum_{j=1}^d (r_j - s_j) \int_0^1 \left( \prod_{\substack{i=1 \\ i \neq j}}^d |\lambda_k c_i^k|^{\frac{1}{4}} \bar{\eta}_i(|\lambda_k c_i^k|^{\frac{1}{2}}(\tilde{y}_i + t(r_i - s_i))) \right. \\
&\quad \left. |\lambda_k c_j^k|^{\frac{3}{4}} \partial_j \bar{\eta}_j(|\lambda_k c_j^k|^{\frac{1}{2}}(\tilde{y}_j + t(r_j - s_j))) \right) dt.
\end{aligned}$$

Thus, since  $f$  and the functions  $(\eta_j)_{j \in \{1, \dots, d\}}$  are Schwartz functions, one can find an on

$(\eta_j)_{j \in \{1, \dots, d\}}$  depending constant  $C_5$  such that

$$\begin{aligned} & \left| \left( \bar{\eta}_{k,0}(\tilde{y}) - \bar{\eta}_{k,0}(\tilde{y} + r - s) \right) \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k c^k}{2}(s + r), \rho_k, \lambda_k, 0, 0 \right) \right| \\ & \leq \|r - s\| \left( \sum_{j=1}^d \prod_{i=1}^d |\lambda_k c_i^k|^{\frac{1}{4}} |\lambda_k c_j^k|^{\frac{1}{2}} \right) \frac{C_5}{(1 + \|r - s\|)^{2d+1}}. \end{aligned}$$

Now, one has the following estimation for  $\|w_k\|_2$ , which is again similar to the above ones:

$$\begin{aligned} \|w_k\|_2^2 &= \int_{\mathbb{R}^d} |w_k(s)|^2 ds \\ &\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\xi(r)| |\eta_{k,0}(\tilde{y})| \|r - s\| \left( \sum_{j=1}^d \prod_{i=1}^d |\lambda_k c_i^k|^{\frac{1}{4}} |\lambda_k c_j^k|^{\frac{1}{2}} \right) \right. \\ &\quad \left. \frac{C_5}{(1 + \|r - s\|)^{2d+1}} dr d\tilde{y} \right)^2 ds \\ &\stackrel{\text{Cauchy-Schwarz}}{\leq} C_5' \left( \sum_{j=1}^d \prod_{i=1}^d |\lambda_k c_i^k|^{\frac{1}{2}} |\lambda_k c_j^k| \right) \int_{\mathbb{R}^{3d}} \frac{|\xi(r)|^2}{(1 + \|r - s\|)^{2d+2}} |\eta_{k,0}(\tilde{y})|^2 \|r - s\|^2 d(r, \tilde{y}, s) \\ &\stackrel{\|\eta_{k,0}\|_{2=1}}{\leq} C_5'' \left( \sum_{j=1}^d \prod_{i=1}^d |\lambda_k c_i^k|^{\frac{1}{2}} |\lambda_k c_j^k| \right) \|\xi\|_2^2, \end{aligned}$$

whereat the constants  $C_5' > 0$  and  $C_5'' > 0$  depend on  $(\eta_j)_{j \in \{1, \dots, d\}}$ . Therefore, for  $\epsilon_k := \left( C_5'' \left( \sum_{j=1}^d \prod_{i=1}^d |\lambda_k c_i^k|^{\frac{1}{2}} |\lambda_k c_j^k| \right) \right)^{\frac{1}{2}}$ , the desired properties  $\epsilon_k \xrightarrow{k \rightarrow \infty} 0$  and

$$\|w_k\|_2 \leq \epsilon_k \|\xi\|_2$$

are fulfilled.

Thus, for those  $f \in \mathcal{S}(\mathbb{R}^{2d+2+p}) \cong \mathcal{S}(G)$  whose Fourier transform in  $[\mathfrak{g}, \mathfrak{g}]$  has a compact support,

$$\left\| \pi_k(f) - \nu_k(p_{G/U}(f)) \right\|_{op} = \sup_{\substack{\xi \in L^2(\mathbb{R}^d) \\ \|\xi\|_2=1}} \left\| (\pi_k(f) - \nu_k(p_{G/U}(f))) (\xi) \right\|_2 \xrightarrow{k \rightarrow \infty} 0.$$

Because of the density in  $L^1(G)$  and thus in  $C^*(G)$  of the set of Schwartz functions  $f \in \mathcal{S}(G)$  whose partial Fourier transform has a compact support, the claim is true for general  $a \in C^*(G)$ .  $\square$

#### 4.6.3 Transition to $(\pi_k^V)_k$

As for every  $k \in \mathbb{N}$  the two representations  $\pi_k$  and  $\pi_k^V$  are equivalent, there exist unitary intertwining operators

$$F_k : \mathcal{H}_{\pi_k^V} \cong L^2(\mathbb{R}^d) \rightarrow \mathcal{H}_{\pi_k} \cong L^2(\mathbb{R}^d) \quad \text{with} \quad F_k \circ \pi_k^V(a) = \pi_k(a) \circ F_k \quad \forall a \in C^*(G).$$

Futhermore, since the limit set  $L((\pi_k^V)_k)$  of the sequence  $(\pi_k^V)_k$  is contained in  $S_{i-1}$ , as discussed in Section 4.1, identifying  $\widehat{G}$  with the set of coadjoint orbits  $\mathfrak{g}^*/G$ , one can restrict an operator field  $\varphi \in CB(S_{i-1})$  to  $L((O_k)_k) = \ell + \mathfrak{u}^\perp$  and obtains an element in  $CB(\ell + \mathfrak{u}^\perp)$ . Thus, as

$$\{\mathcal{F}(a)|_{L((O_k)_k)} \mid a \in C^*(G)\} = C_0(L((O_k)_k)) = C_0(\ell + \mathfrak{u}^\perp),$$



one can define the  $*$ -isomorphism

$$\tau : C_0(\mathbb{R}^{2d}) \cong C_0(\ell + \mathbf{u}^\perp) \rightarrow C^*(G/U, \chi_\ell) \cong C^*(\mathbb{R}^{2d}), \quad \mathcal{F}(a)|_{L((O_k)_k)} \mapsto p_{G/U}(a).$$

Now, define  $\tilde{\nu}_k$  as

$$\tilde{\nu}_k(\varphi) = F_k^* \circ (\nu_k \circ \tau)(\varphi|_{L((O_k)_k)}) \circ F_k \quad \forall \varphi \in CB(S_{i-1}).$$

Since the image of  $\nu_k$  is in  $\mathcal{B}(L^2(\mathbb{R}^d))$  and  $F_k$  is an intertwining operator and thus bounded, the image of  $\tilde{\nu}_k$  is contained in  $\mathcal{B}(L^2(\mathbb{R}^d))$  as well.

Moreover, the operator  $\tilde{\nu}_k$  is bounded: From the boundedness of  $\nu_k$  (Proposition 4.1) and using that  $\tau$  is an isomorphism, one gets for every  $\varphi \in CB(S_{i-1})$

$$\begin{aligned} \|\tilde{\nu}_k(\varphi)\|_{op} &= \|F_k^* \circ (\nu_k \circ \tau)(\varphi|_{L((O_k)_k)}) \circ F_k\|_{op} \\ &\leq \|(\nu_k \circ \tau)(\varphi|_{L((O_k)_k)})\|_{op} \\ &\leq \|\tau(\varphi|_{L((O_k)_k)})\|_{C^*(\mathbb{R}^{2d})} \\ &\leq \|(\varphi|_{L((O_k)_k)})\|_\infty \leq \|\varphi\|_{S_{i-1}}. \end{aligned}$$

The involutivity of  $\tilde{\nu}_k$  follows from the involutivity of  $\tau$  and  $\nu_k$  (Proposition 4.1).

Finally, the demanded convergence of Condition 3(b) can also be shown: With the above stated equivalence of the representations  $\pi_k$  and  $\pi_k^V$ , one gets

$$\begin{aligned} \|\pi_k^V(a) - \tilde{\nu}_k(\mathcal{F}(a)|_{S_{i-1}})\|_{op} &= \|F_k^* \circ \pi_k(a) \circ F_k - F_k^* \circ (\nu_k \circ \tau)(\mathcal{F}(a)|_{L((O_k)_k)}) \circ F_k\|_{op} \\ &= \|F_k^* \circ \pi_k(a) \circ F_k - F_k^* \circ \nu_k(p_{G/U}(a)) \circ F_k\|_{op} \\ &= \|F_k^* \circ (\pi_k(a) - \nu_k(p_{G/U}(a))) \circ F_k\|_{op} \\ &\leq \|\nu_k(p_{G/U}(a)) - \pi_k(a)\|_{op} \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Therefore, the representations  $(\pi_k^V)_k$  fulfill Property 3(b) and the conditions of the theorem are thus proved.

## 4.7 Third case

In the third and last case  $\lambda \neq 0$  and there exists  $1 \leq m < d$  such that  $c_j \neq 0$  for every  $j \in \{1, \dots, m\}$  and  $c_j = 0$  for every  $j \in \{m+1, \dots, d\}$ .

This means that

$$\langle \ell_k, [X_j^k, Y_j^k] \rangle = c_j^k \lambda_k \xrightarrow{k \rightarrow \infty} c_j \lambda = 0 \iff j \in \{m+1, \dots, d\}.$$

In this case  $\mathfrak{p} := \text{span}\{X_{m+1}, \dots, X_d, Y_1, \dots, Y_d, T, Z, A_1, \dots, A_p\}$  is a polarization for  $\ell$ .

Moreover, for  $\tilde{\mathfrak{p}}_k := \text{span}\{X_{m+1}^k, \dots, X_d^k, Y_1^k, \dots, Y_d^k, T_k, Z_k, A_1^k, \dots, A_p^k\}$ , one has  $\tilde{\mathfrak{p}}_k \xrightarrow{k \rightarrow \infty} \mathfrak{p}$ .

Let  $P := \exp(\mathfrak{p})$  and  $\tilde{P}_k := \exp(\tilde{\mathfrak{p}}_k)$ .

### 4.7.1 Convergence of $(\pi_k)_k$ in $\hat{G}$

Let

$$(x)_\infty = (\dot{x}, \ddot{x})_\infty \quad \text{with} \quad (\dot{x})_\infty := (x_1, \dots, x_m)_\infty \quad \text{and} \quad (\ddot{x})_\infty := (x_{m+1}, \dots, x_d)_\infty$$

and analogously

$$(y)_\infty = (\dot{y}, \ddot{y})_\infty \quad \text{with} \quad (\dot{y})_\infty := (y_1, \dots, y_m)_\infty \quad \text{and} \quad (\ddot{y})_\infty := (y_{m+1}, \dots, y_d)_\infty.$$

Moreover, as in Chapter 4.3 above, let

$$(a)_\infty = (\dot{a}, \ddot{a})_\infty \quad \text{with} \quad (\dot{a})_\infty = (a_1, \dots, a_{\tilde{p}})_\infty \quad \text{and} \quad (\ddot{a})_\infty = (a_{\tilde{p}+1}, \dots, a_p)_\infty$$

and let

$$(g)_\infty = (\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, t, z, \dot{a}, \ddot{a})_\infty = (x, y, t, z, a)_\infty = (x, h)_\infty.$$

Now, let  $\ddot{\alpha} := (\alpha_{m+1}, \dots, \alpha_d) \in \mathbb{R}^{d-m}$  and  $\ddot{\beta} := (\beta_{m+1}, \dots, \beta_d) \in \mathbb{R}^{d-m}$ , consider  $\ddot{\alpha}$  and  $\ddot{\beta}$  as elements of  $\mathbb{R}^d$  identifying them with  $(0, \dots, 0, \alpha_{m+1}, \dots, \alpha_d)$  and  $(0, \dots, 0, \beta_{m+1}, \dots, \beta_d)$ , respectively and let

$$\tilde{\pi} := \tilde{\pi}_{\ddot{\alpha}, \ddot{\beta}} := \text{ind}_P^G \chi_{\ell + \ell_{\ddot{\alpha}, \ddot{\beta}}}.$$

Then, for a function  $\dot{\xi}$  in the representation space  $\mathcal{H}_{\tilde{\pi}} = L^2(G/P, \chi_{\ell + \ell_{\ddot{\alpha}, \ddot{\beta}}})$  of  $\tilde{\pi}$  and for  $s_1, \dots, s_m \in \mathbb{R}$ ,  $(\dot{s})_\infty = (s_1, \dots, s_m)_\infty \in \text{span}\{X_1\} \times \dots \times \text{span}\{X_m\}$  and  $\dot{c} = (c_1, \dots, c_m)$ , letting  $\rho := \langle \ell, T \rangle$  one has similarly as in (5):

$$\tilde{\pi}((g)_\infty) \dot{\xi}((\dot{s})_\infty) = e^{2\pi i(-t\rho - z\lambda + \lambda \dot{c}((\dot{s})_\infty - \frac{1}{2}(\dot{x})_\infty)(\dot{y})_\infty))} e^{-2\pi i(\ddot{\alpha}(\ddot{x})_\infty + \ddot{\beta}(\ddot{y})_\infty)} \dot{\xi}((\dot{s} - \dot{x})_\infty),$$

since  $\ell(Y_j) = \ell(X_j) = 0$  for all  $j \in \{1, \dots, d\}$ .

From now on again, most of the time,  $G$  will be identified with  $\mathbb{R}^{2d+2+p}$ .

Define for  $\ddot{s} = (s_{m+1}, \dots, s_d) \in \text{span}\{X_{m+1}\} \times \dots \times \text{span}\{X_d\} \cong \mathbb{R}^{d-m}$

$$\ddot{\eta}_{k, \ddot{\alpha}, \ddot{\beta}}(\ddot{s}) := e^{2\pi i \ddot{\alpha} \ddot{s}} \prod_{j=m+1}^d |\lambda_k c_j^k|^{\frac{1}{4}} \eta_j \left( |\lambda_k c_j^k|^{\frac{1}{2}} \left( s_j + \frac{\beta_j}{\lambda_k c_j^k} \right) \right)$$

and furthermore for  $\dot{\xi} \in \mathcal{H}_{\tilde{\pi}} = L^2(G/P, \chi_{\ell + \ell_{\ddot{\alpha}, \ddot{\beta}}}) \cong L^2(\mathbb{R}^m)$  and  $s = (\dot{s}, \ddot{s})$  in  $(\text{span}\{X_1\} \times \dots \times \text{span}\{X_m\}) \times (\text{span}\{X_{m+1}\} \times \dots \times \text{span}\{X_d\}) \cong \mathbb{R}^m \times \mathbb{R}^{d-m}$

$$\xi_k(s) := \dot{\xi}(\dot{s}) \ddot{\eta}_{k, \ddot{\alpha}, \ddot{\beta}}(\ddot{s}).$$

Then, as above in the second case, the coefficient functions  $c_{\ddot{\alpha}, \ddot{\beta}}^k$  defined by

$$c_{\ddot{\alpha}, \ddot{\beta}}^k(g) := \langle \pi_k(g) \xi_k, \xi_k \rangle \quad \forall g \in G \cong \mathbb{R}^{2d+2+p}$$

converge uniformly on compacta to  $c_{\ddot{\alpha}, \ddot{\beta}}$  which in turn is defined by

$$c_{\ddot{\alpha}, \ddot{\beta}}(g) := \langle \tilde{\pi}_{\ddot{\alpha}, \ddot{\beta}}(g) \dot{\xi}, \dot{\xi} \rangle \quad \forall g \in G \cong \mathbb{R}^{2d+2+p}.$$

#### 4.7.2 Definition of the $\nu_k$ 's

For  $0 \neq \eta' \in L^2(\tilde{P}_k/P_k, \chi_{\ell_k}) \cong L^2(\mathbb{R}^{d-m})$  let

$$P_{\eta'} : L^2(\mathbb{R}^{d-m}) \rightarrow \mathbb{C}\eta', \quad \xi \mapsto \eta' \langle \xi, \eta' \rangle.$$

Then  $P_{\eta'}$  is the orthogonal projection onto the space  $\mathbb{C}\eta'$ .

Define now for  $k \in \mathbb{N}$  and  $h \in C^*(G/U, \chi_\ell)$  the linear operator

$$\nu_k(h) := \int_{\mathbb{R}^{2(d-m)}} \pi_{\ell + (\tilde{x}, \tilde{y})}(h) \otimes P_{\ddot{\eta}_{k, \tilde{x}, \tilde{y}}} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|},$$

whereat  $\pi_{\ell+(\tilde{x},\tilde{y})}$  is defined as  $\text{ind}_P^G \chi_{\ell+(\tilde{x},\tilde{y})}$  for an element  $\ell + (\tilde{x},\tilde{y})$  located in  $\ell + ((\text{span}\{X_{m+1}\} \times \dots \times \text{span}\{X_d\}) \times (\text{span}\{Y_{m+1}\} \times \dots \times \text{span}\{Y_d\}))^* \cong \ell + \mathbb{R}^{2(d-m)}$ .

Thus, for  $L^2(\mathbb{R}^d) \ni \xi = \sum_{i=1}^{\infty} \dot{\xi}_i \otimes \ddot{\xi}_i$  with  $\dot{\xi}_i \in L^2(\mathbb{R}^m)$  and  $\ddot{\xi}_i \in L^2(\mathbb{R}^{d-m})$  for all  $i \in \mathbb{N}$ , one has

$$\nu_k(h)(\xi) := \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \pi_{\ell+(\tilde{x},\tilde{y})}(h)(\dot{\xi}_i) \otimes P_{\tilde{\eta}_k, \tilde{x}, \tilde{y}}(\ddot{\xi}_i) \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|}.$$

**Proposition 4.3.**

1. For every  $k \in \mathbb{N}$  and  $h \in \mathcal{S}(G/U, \chi_\ell)$  the integral defining  $\nu_k(h)$  converges in the operator norm.
2. The operator  $\nu_k(h)$  is compact and  $\|\nu_k(h)\|_{op} \leq \|h\|_{C^*(G/U, \chi_\ell)}$ .
3.  $\nu_k$  is involutive.

Proof:

Let  $\mathcal{K} = \mathcal{K}(L^2(\mathbb{R}^m))$  be the  $C^*$ -algebra of the compact operators on the Hilbert space  $L^2(\mathbb{R}^m)$  and  $C_0(\mathbb{R}^{2(d-m)}, \mathcal{K})$  the  $C^*$ -algebra of all continuous mappings from  $\mathbb{R}^{2(d-m)}$  into  $\mathcal{K}$  vanishing at infinity.

Define for  $\varphi \in C_0(\mathbb{R}^{2(d-m)}, \mathcal{K})$  and  $k \in \mathbb{N}$  the linear operator

$$\mu_k(\varphi) := \int_{\mathbb{R}^{2(d-m)}} \varphi(\tilde{x}, \tilde{y}) \otimes P_{\tilde{\eta}_k, \tilde{x}, \tilde{y}} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|}$$

on  $L^2(\mathbb{R}^d)$ . Then, as  $\mathcal{F}(h) \in C_0(\mathbb{R}^{2(d-m)}, \mathcal{K})$  for  $h \in C^*(G/U, \chi_\ell)$ ,

$$\nu_k(h) = \mu_k(\mathcal{F}(h)).$$

1. Since  $\mathcal{F}(h) \in \mathcal{S}(\mathbb{R}^{2(d-m)}, \mathcal{K})$  for  $h \in \mathcal{S}(G/U, \chi_\ell)$  and since

$$\|\mu_k(\varphi)\|_{op} \leq \int_{\mathbb{R}^{2(d-m)}} \|\varphi(\tilde{x}, \tilde{y})\|_{op} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|}$$

for every  $\varphi \in \mathcal{S}(\mathbb{R}^{2(d-m)}, \mathcal{K})$  and  $k \in \mathbb{N}$ , the first assertion follows immediately.

2. As  $p_{G/U}$  is surjective from the space  $\mathcal{S}(G)$  onto the space  $\mathcal{S}(G/U, \chi_\ell)$ , for every  $h \in \mathcal{S}(G/U, \chi_\ell) \cong \mathcal{S}(\mathbb{R}^{2d})$  there exists a function  $f \in \mathcal{S}(G) \cong \mathcal{S}(\mathbb{R}^{2d+2+p})$  such that  $h = p_{G/U}(f)$  and, as shown in the second case, for  $\tilde{g} = (\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, 0, 0, 0, 0) \in G/U \cong \mathbb{R}^{2d}$  one has

$$h_\infty(\tilde{g}) = \hat{f}_\infty^{5,6,7,8}(\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, \rho, \lambda, 0, 0), \quad (9)$$

where again  $h_\infty = h((\cdot)_\infty)$  and  $f_\infty = f((\cdot)_\infty)$ .

Now, let  $s_1, \dots, s_m \in \mathbb{R}$  and  $\dot{s} = (s_1, \dots, s_m)_\infty = \sum_{j=1}^m s_j X_j$  and moreover, let  $(g)_\infty = (\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, t, z, \dot{a}, \ddot{a})_\infty = (x, y, t, z, a)_\infty = (x, h)_\infty$  with  $\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, \dot{a}, \ddot{a}$  and  $\dot{c}$  as above. Then, one gets for  $\dot{\xi}_i \in L^2(\mathbb{R}^m)$

$$\pi_{\ell+(\tilde{x},\tilde{y})}((g)_\infty) \dot{\xi}_i((\dot{s})_\infty) = e^{2\pi i(-t\rho - z\lambda + \lambda \dot{c}((\dot{s})_\infty - \frac{1}{2}(\dot{x})_\infty)(\dot{y})_\infty) - \langle \tilde{x}, (\ddot{x})_\infty \rangle - \langle \tilde{y}, (\ddot{y})_\infty \rangle} \dot{\xi}_i((\dot{s} - \dot{x})_\infty).$$

Using the Equality (9) and identifying  $G$  with  $\mathbb{R}^{2d+2+p}$ , one gets for a function  $h = p_{G/U}(f) \in \mathcal{S}(G/U, \chi_\ell) \cong \mathcal{S}(\mathbb{R}^{2d})$  with  $f \in \mathcal{S}(G) \cong \mathcal{S}(\mathbb{R}^{2d+2+p})$

$$\begin{aligned}
\pi_{\ell+(\tilde{x}, \tilde{y})}(h) \dot{\xi}_i(\dot{s}) &= \int_{\mathbb{R}^{2d}} (p_{G/U}(f))_\infty(\tilde{g}) \pi_{\ell+(\tilde{x}, \tilde{y})}(\tilde{g}) \dot{\xi}_i(\dot{s}) d\tilde{g} \\
&= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2+\bar{p}+(p-\bar{p})}} f_\infty(\tilde{g}u) \chi_\ell(u) du \pi_{\ell+(\tilde{x}, \tilde{y})}(\tilde{g}) \dot{\xi}_i(\dot{s}) d\tilde{g} \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{\bar{p}} \times \mathbb{R}^{p-\bar{p}}} f_\infty(\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, t, z, \dot{a}, \ddot{a}) e^{2\pi i(\lambda \dot{c}((s-\frac{1}{2}\dot{x})\dot{y}) - \langle \tilde{x}, \ddot{x} \rangle - \langle \tilde{y}, \ddot{y} \rangle)} \\
&\quad e^{-2\pi i(t\rho + z\lambda)} \dot{\xi}_i(\dot{s} - \dot{x}) d(\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, t, z, \dot{a}, \ddot{a}) \\
&= \int_{\mathbb{R}^{2m}} \hat{f}_\infty^{2,4,5,6,7,8}(\dot{x}, \tilde{x}, \dot{y}, \tilde{y}, \rho, \lambda, 0, 0) e^{2\pi i\lambda \dot{c}((s-\frac{1}{2}\dot{x})\dot{y})} \dot{\xi}_i(\dot{s} - \dot{x}) d(\dot{x}, \dot{y}) \\
&= \int_{\mathbb{R}^{2m}} \hat{h}_\infty^{2,4}(\dot{x}, \tilde{x}, \dot{y}, \tilde{y}) e^{2\pi i\lambda \dot{c}((s-\frac{1}{2}\dot{x})\dot{y})} \dot{\xi}_i(\dot{s} - \dot{x}) d(\dot{x}, \dot{y}) \\
&= \int_{\mathbb{R}^m} \hat{h}_\infty^{2,3,4}\left(\dot{x}, \tilde{x}, \lambda \dot{c}\left(\frac{1}{2}\dot{x} - \dot{s}\right), \tilde{y}\right) \dot{\xi}_i(\dot{s} - \dot{x}) d\dot{x}. \tag{10}
\end{aligned}$$

Regard now the second factor  $P_{\tilde{\eta}_{k,\tilde{x},\tilde{y}}}$  of the tensor product:  
As in the second case above, define

$$\ddot{\eta}_{k,\tilde{\beta}}(\ddot{s}) := \prod_{j=m+1}^d |\lambda_k c_j^k|^{\frac{1}{4}} \eta_j \left( |\lambda_k c_j^k|^{\frac{1}{2}} \left( s_j + \frac{\beta_j}{\lambda_k c_j^k} \right) \right).$$

Then

$$\ddot{\eta}_{k,\ddot{\alpha},\ddot{\beta}}(\ddot{s}) = e^{2\pi i\ddot{\alpha}\ddot{s}} \ddot{\eta}_{k,\ddot{\beta}}(\ddot{s})$$

and therefore with  $\ddot{\xi}_i \in L^2(\mathbb{R}^{d-m})$

$$\begin{aligned}
P_{\tilde{\eta}_{k,\tilde{x},\tilde{y}}}(\ddot{\xi}_i)(\ddot{s}) &= \langle \ddot{\xi}_i, \ddot{\eta}_{k,\tilde{x},\tilde{y}} \rangle \ddot{\eta}_{k,\tilde{x},\tilde{y}}(\ddot{s}) \\
&= \left( \int_{\mathbb{R}^{d-m}} \ddot{\xi}_i(\ddot{r}) \overline{\ddot{\eta}_{k,\tilde{x},\tilde{y}}}(\ddot{r}) d\ddot{r} \right) \ddot{\eta}_{k,\tilde{x},\tilde{y}}(\ddot{s}) \\
&= \left( \int_{\mathbb{R}^{d-m}} \ddot{\xi}_i(\ddot{r}) e^{-2\pi i\tilde{x}\ddot{r}} \overline{\ddot{\eta}_{k,\tilde{y}}}(\ddot{r}) d\ddot{r} \right) e^{2\pi i\tilde{x}\ddot{s}} \ddot{\eta}_{k,\tilde{y}}(\ddot{s}) \\
&= \int_{\mathbb{R}^{d-m}} \ddot{\xi}_i(\ddot{r}) e^{2\pi i\tilde{x}(\ddot{s}-\ddot{r})} \overline{\ddot{\eta}_{k,\tilde{y}}}(\ddot{r}) d\ddot{r} \ddot{\eta}_{k,\tilde{y}}(\ddot{s}).
\end{aligned}$$

Joining together the calculation above and the one for the first factor of the tensor product (10), one gets for  $\mathcal{S}(\mathbb{R}^d) \ni \xi = \sum_{i=1}^{\infty} \dot{\xi}_i \otimes \ddot{\xi}_i$  with  $\dot{\xi}_i \in \mathcal{S}(\mathbb{R}^m)$  and  $\ddot{\xi}_i \in \mathcal{S}(\mathbb{R}^{d-m})$  for all  $i \in \mathbb{N}$ ,

$h \in \mathcal{S}(\mathbb{R}^{2d})$  and  $s = (\dot{s}, \ddot{s}) \in \mathbb{R}^d$

$$\begin{aligned}
& \nu_k(h)(\xi)(s) \\
&= \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \pi_{\ell+(\tilde{x}, \tilde{y})}(h)(\dot{\xi}_i)(\dot{s}) \cdot P_{\tilde{\eta}_{k, \tilde{x}, \tilde{y}}}(\ddot{\xi}_i)(\ddot{s}) \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \\
&= \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \left( \int_{\mathbb{R}^m} \hat{h}_{\infty}^{2,3,4} \left( \dot{x}, \tilde{x}, \lambda \dot{c} \left( \frac{1}{2} \dot{x} - \dot{s} \right), \tilde{y} \right) \dot{\xi}_i(\dot{s} - \dot{x}) d\dot{x} \right) \\
&\quad \cdot \left( \int_{\mathbb{R}^{d-m}} \ddot{\xi}_i(\ddot{r}) e^{2\pi i \tilde{x}(\ddot{s} - \ddot{r})} \overline{\tilde{\eta}_{k, \tilde{y}}}(\ddot{r}) d\ddot{r} \ddot{\eta}_{k, \tilde{y}}(\ddot{s}) \right) \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \\
&= \sum_{i=1}^{\infty} \int_{\mathbb{R}^{2(d-m)}} \int_{\mathbb{R}^{d-m}} \int_{\mathbb{R}^m} \hat{h}_{\infty}^{2,3,4} \left( \dot{x}, \tilde{x}, \lambda \dot{c} \left( \frac{1}{2} \dot{x} - \dot{s} \right), \tilde{y} \right) \dot{\xi}_i(\dot{s} - \dot{x}) \\
&\quad \ddot{\xi}_i(\ddot{r}) e^{2\pi i \tilde{x}(\ddot{s} - \ddot{r})} \overline{\tilde{\eta}_{k, \tilde{y}}}(\ddot{r}) d\dot{x} d\ddot{r} \ddot{\eta}_{k, \tilde{y}}(\ddot{s}) \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \\
&= \sum_{i=1}^{\infty} \int_{\mathbb{R}^{d-m}} \int_{\mathbb{R}^{d-m}} \int_{\mathbb{R}^m} \hat{h}_{\infty}^{3,4} \left( \dot{x}, \ddot{s} - \ddot{r}, \lambda \dot{c} \left( \frac{1}{2} \dot{x} - \dot{s} \right), \tilde{y} \right) \dot{\xi}_i(\dot{s} - \dot{x}) \\
&\quad \ddot{\xi}_i(\ddot{r}) \overline{\tilde{\eta}_{k, \tilde{y}}}(\ddot{r}) d\dot{x} d\ddot{r} \ddot{\eta}_{k, \tilde{y}}(\ddot{s}) \frac{d\tilde{y}}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \\
&= \sum_{i=1}^{\infty} \int_{\mathbb{R}^{d-m}} \int_{\mathbb{R}^{d-m}} \int_{\mathbb{R}^m} \hat{h}_{\infty}^{3,4} \left( \dot{s} - \dot{x}, \ddot{s} - \ddot{r}, \frac{\lambda}{2} \dot{c}(-\dot{x} - \dot{s}), \tilde{y} \right) \dot{\xi}_i(\dot{x}) \\
&\quad \ddot{\xi}_i(\ddot{r}) \overline{\tilde{\eta}_{k, \tilde{y}}}(\ddot{r}) d\dot{x} d\ddot{r} \ddot{\eta}_{k, \tilde{y}}(\ddot{s}) \frac{d\tilde{y}}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-m}} \hat{h}_{\infty}^{3,4} \left( \dot{s} - \dot{x}, \ddot{s} - \ddot{r}, \frac{\lambda}{2} \dot{c}(-\dot{x} - \dot{s}), \tilde{y} \right) \overline{\tilde{\eta}_{k, \tilde{y}}}(\ddot{r}) \ddot{\eta}_{k, \tilde{y}}(\ddot{s}) \frac{d\tilde{y}}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \\
&\quad \xi(\dot{x}, \ddot{r}) d(\dot{x}, \ddot{r}). \tag{11}
\end{aligned}$$

Therefore, the kernel function

$$h_K((\dot{s}, \ddot{s}), (\dot{x}, \ddot{r})) := \int_{\mathbb{R}^{d-m}} \hat{h}_{\infty}^{3,4} \left( \dot{s} - \dot{x}, \ddot{s} - \ddot{r}, \frac{\lambda}{2} \dot{c}(-\dot{x} - \dot{s}), \tilde{y} \right) \overline{\tilde{\eta}_{k, \tilde{y}}}(\ddot{r}) \ddot{\eta}_{k, \tilde{y}}(\ddot{s}) \frac{d\tilde{y}}{\prod_{j=m+1}^d |\lambda_k c_j^k|}$$

of  $\nu_k(h)$  is contained in  $\mathcal{S}(\mathbb{R}^{2d})$  and thus  $\nu_k(h)$  is a compact operator for  $h \in \mathcal{S}(\mathbb{R}^{2d}) \cong \mathcal{S}(G/U, \chi_{\ell})$  and with the density of  $\mathcal{S}(G/U, \chi_{\ell})$  in  $C^*(G/U, \chi_{\ell})$ , it is compact for every  $h \in C^*(G/U, \chi_{\ell})$ .

Now, it is shown that for every  $\varphi \in C_0(\mathbb{R}^{2(d-m)}, \mathcal{K})$

$$\|\mu_k(\varphi)\|_{\text{op}} \leq \|\varphi\|_{\infty} := \sup_{(\tilde{x}, \tilde{y}) \in \mathbb{R}^{2(d-m)}} \|\varphi(\tilde{x}, \tilde{y})\|_{\text{op}}.$$

For this, for any  $\psi \in L^2(\mathbb{R}^d)$ , define

$$f_{\psi,k}(\tilde{x}, \tilde{y})(\dot{s}) := \int_{\mathbb{R}^{d-m}} \psi(\dot{s}, \ddot{s}) \overline{\eta}_{k,\tilde{x},\tilde{y}}(\ddot{s}) d\ddot{s} \quad \forall (\tilde{x}, \tilde{y}) \in \mathbb{R}^{2(d-m)} \quad \forall \dot{s} \in \mathbb{R}^m.$$

Then, as

$$\mathbb{I}_{L^2(\mathbb{R}^{d-m})} = \int_{\mathbb{R}^{2(d-m)}} P_{\eta_{k,\tilde{x},\tilde{y}}} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|},$$

one gets the identity

$$\|\psi\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^{2(d-m)}} \|f_{\psi,k}(\tilde{x}, \tilde{y})\|_{L^2(\mathbb{R}^m)}^2 \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|}. \quad (12)$$

Now, for  $\xi, \psi \in L^2(\mathbb{R}^d)$

$$\begin{aligned} & \left| \langle \mu_k(\varphi)\xi, \psi \rangle_{L^2(\mathbb{R}^d)} \right| \\ &= \left| \int_{\mathbb{R}^{2(d-m)}} \langle (\varphi(\tilde{x}, \tilde{y}) \otimes P_{\eta_{k,\tilde{x},\tilde{y}}})\xi, \psi \rangle_{L^2(\mathbb{R}^d)} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \right| \\ &= \left| \int_{\mathbb{R}^{2(d-m)}} \langle (\varphi(\tilde{x}, \tilde{y}) \otimes \mathbb{I}_{L^2(\mathbb{R}^{d-m})}) \circ (\mathbb{I}_{L^2(\mathbb{R}^m)} \otimes P_{\eta_{k,\tilde{x},\tilde{y}}})\xi, (\mathbb{I}_{L^2(\mathbb{R}^m)} \otimes P_{\eta_{k,\tilde{x},\tilde{y}}})\psi \rangle_{L^2(\mathbb{R}^d)} \right. \\ & \quad \left. \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \right| \\ &= \left| \int_{\mathbb{R}^{2(d-m)}} \langle \varphi(\tilde{x}, \tilde{y}) f_{\xi,k}(\tilde{x}, \tilde{y}), f_{\psi,k}(\tilde{x}, \tilde{y}) \rangle_{L^2(\mathbb{R}^m)} \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \right| \\ &\stackrel{\text{Cauchy-Schwarz}}{\leq} \left( \int_{\mathbb{R}^{2(d-m)}} \|\varphi(\tilde{x}, \tilde{y}) f_{\xi,k}(\tilde{x}, \tilde{y})\|_{L^2(\mathbb{R}^m)}^2 \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \right)^{\frac{1}{2}} \\ & \quad \left( \int_{\mathbb{R}^{2(d-m)}} \|f_{\psi,k}(\tilde{x}, \tilde{y})\|_{L^2(\mathbb{R}^m)}^2 \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \right)^{\frac{1}{2}} \\ &\stackrel{(12)}{\leq} \sup_{(\tilde{x}, \tilde{y}) \in \mathbb{R}^{2(d-m)}} \|\varphi(\tilde{x}, \tilde{y})\|_{\text{op}} \left( \int_{\mathbb{R}^{2(d-m)}} \|f_{\xi,k}(\tilde{x}, \tilde{y})\|_{L^2(\mathbb{R}^m)}^2 \frac{d(\tilde{x}, \tilde{y})}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \right)^{\frac{1}{2}} \|\psi\|_{L^2(\mathbb{R}^d)} \\ &\stackrel{(12)}{\leq} \|\varphi\|_{\infty} \|\xi\|_{L^2(\mathbb{R}^d)} \|\psi\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Hence, for every  $h \in C^*(G/U, \chi_\ell)$ ,

$$\|\nu_k(h)\|_{\text{op}} = \|\mu_k(\mathcal{F}(h))\|_{\text{op}} \leq \|\mathcal{F}(h)\|_{\infty} = \|h\|_{C^*(G/U, \chi_\ell)}.$$

3. To show that  $\nu_k$  is involutive is as straightforward as in the second case.

□

The demanded convergence of Condition 3(b) remains to be shown:

#### 4.7.3 Theorem - Third Case

##### Theorem 4.4.

For  $a \in C^*(G)$

$$\lim_{k \rightarrow \infty} \|\pi_k(a) - \nu_k(p_{G/U}(a))\|_{op} = 0.$$

Proof:

Let  $f \in \mathcal{S}(G) \cong \mathcal{S}(\mathbb{R}^{2d+2+p})$  such that its Fourier transform in  $[\mathfrak{g}, \mathfrak{g}]$  has a compact support on  $G \cong \mathbb{R}^{2d+2+p}$ . In the setting of this third case, this means that  $\hat{f}^{6,7}$  has a compact support in  $G$  (see Theorem 4.2).

Now, identify  $G$  with  $\mathbb{R}^{2d+2+p}$  again, let  $\xi \in L^2(\mathbb{R}^d)$  and  $s = (s_1, \dots, s_d) = (\dot{s}, \ddot{s})$  be located in  $\mathbb{R}^m \times \mathbb{R}^{d-m} \cong \mathbb{R}^d$  and define

$$\ddot{\eta}_{k,0}(\ddot{s}) := \prod_{j=m+1}^d |\lambda_k c_j^k|^{\frac{1}{4}} \eta_j \left( |\lambda_k c_j^k|^{\frac{1}{2}}(s_j) \right).$$

Moreover, let  $\dot{c} = (c_1, \dots, c_m)$ ,  $\ddot{c} = (c_{m+1}, \dots, c_d) = (0, \dots, 0)$ ,  $\dot{c}^k = (c_1^k, \dots, c_m^k)$  and  $\ddot{c}^k = (c_{m+1}^k, \dots, c_d^k)$ .

As in the second case, the expression  $(\pi_k(f) - \nu_k(p_{G/U}(f)))$  is now going to be regarded, composed into several parts and then estimated: For this, Equation (6) from Chapter 4.4 will be used again but its notation needs to be adapted:

$$\begin{aligned} \pi_k(f)(s) &\stackrel{(6)}{=} \int_{\mathbb{R}^d} \hat{f}_k^{2,3,4,5,6} \left( s - r, -\frac{\lambda_k \dot{c}^k}{2}(s + r), \rho_k, \lambda_k, 0, 0 \right) \xi(r) dr \\ &= \int_{\mathbb{R}^d} \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s} + \ddot{r}), \rho_k, \lambda_k, 0, 0 \right) \xi(\dot{r}, \ddot{r}) d(\dot{r}, \ddot{r}). \end{aligned}$$

Using the above equation, (11) and the fact that  $p_{G/U}(f) = \hat{f}^{5,6,7,8}(\cdot, \cdot, \cdot, \cdot, \rho, \lambda, 0, 0)$ , one gets

$$\begin{aligned}
& (\pi_k(f) - \nu_k(p_{G/U}(f)))\xi(s) \\
(11) \quad & \int_{\mathbb{R}^d} \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s} + \ddot{r}), \rho_k, \lambda_k, 0, 0 \right) \xi(\dot{r}, \ddot{r}) d(\dot{r}, \ddot{r}) \\
& - \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-m}} \widehat{p_{G/U}(f)}_\infty^{3,4} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, \frac{\lambda}{2} \dot{c}(-\dot{r} - \dot{s}), \tilde{y} \right) \bar{\eta}_{k,\tilde{y}}(\ddot{r}) \ddot{\eta}_{k,\tilde{y}}(\ddot{s}) \frac{d\tilde{y}}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \\
& \quad \xi(\dot{r}, \ddot{r}) d(\dot{r}, \ddot{r}) \\
\| \bar{\eta}_{k,0} \|_{2=1} \quad & \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-m}} \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s} + \ddot{r}), \rho_k, \lambda_k, 0, 0 \right) \\
& \quad \bar{\eta}_{k,0}(\tilde{y}) \ddot{\eta}_{k,0}(\tilde{y}) d\tilde{y} \xi(\dot{r}, \ddot{r}) d(\dot{r}, \ddot{r}) \\
& - \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-m}} \hat{f}_\infty^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, \frac{\lambda}{2} \dot{c}(-\dot{r} - \dot{s}), \tilde{y}, \rho, \lambda, 0, 0 \right) \bar{\eta}_{k,\tilde{y}}(\ddot{r}) \ddot{\eta}_{k,\tilde{y}}(\ddot{s}) \frac{d\tilde{y}}{\prod_{j=m+1}^d |\lambda_k c_j^k|} \\
& \quad \xi(\dot{r}, \ddot{r}) d(\dot{r}, \ddot{r}) \\
& = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-m}} \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s} + \ddot{r}), \rho_k, \lambda_k, 0, 0 \right) \\
& \quad \bar{\eta}_{k,0}(\tilde{y}) \ddot{\eta}_{k,0}(\tilde{y}) d\tilde{y} \xi(\dot{r}, \ddot{r}) d(\dot{r}, \ddot{r}) \\
& - \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-m}} \hat{f}_\infty^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda}{2} \dot{c}(\dot{s} + \dot{r}), \lambda_k \ddot{c}^k(\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) \\
& \quad \bar{\eta}_{k,0}(\tilde{y} + \ddot{r} - \ddot{s}) \ddot{\eta}_{k,0}(\tilde{y}) d\tilde{y} \xi(\dot{r}, \ddot{r}) d(\dot{r}, \ddot{r}).
\end{aligned}$$

Similar as for the second case, functions  $q_k$ ,  $u_k$ ,  $v_k$ ,  $o_k$ ,  $n_k$  and  $w_k$  are going to be defined in order to divide the above integrals into six parts:

$$\begin{aligned}
q_k(s, \tilde{y}) \quad & := \int_{\mathbb{R}^d} \xi(\dot{r}, \ddot{r}) \bar{\eta}_{k,0}(\tilde{y} + \ddot{r} - \ddot{s}) \\
& \quad \left( \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s} + \ddot{r}), \rho_k, \lambda_k, 0, 0 \right) \right. \\
& \quad \left. - \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s} + \ddot{r}), \rho, \lambda_k, 0, 0 \right) \right) d(\dot{r}, \ddot{r}),
\end{aligned}$$

$$\begin{aligned}
u_k(s, \tilde{y}) \quad & := \int_{\mathbb{R}^d} \xi(\dot{r}, \ddot{r}) \bar{\eta}_{k,0}(\tilde{y} + \ddot{r} - \ddot{s}) \\
& \quad \left( \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s} + \ddot{r}), \rho, \lambda_k, 0, 0 \right) \right. \\
& \quad \left. - \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s} + \ddot{r}), \rho, \lambda, 0, 0 \right) \right) d(\dot{r}, \ddot{r}),
\end{aligned}$$



$$\begin{aligned}
v_k(s, \tilde{y}) &:= \int_{\mathbb{R}^d} \xi(\dot{r}, \ddot{r}) \bar{\eta}_{k,0}(\tilde{y} + \ddot{r} - \ddot{s}) \\
&\quad \left( \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s} + \ddot{r}), \rho, \lambda, 0, 0 \right) \right. \\
&\quad \left. - \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), \lambda_k \ddot{c}^k(\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) \right) d(\dot{r}, \ddot{r}),
\end{aligned}$$

$$\begin{aligned}
o_k(s, \tilde{y}) &:= \int_{\mathbb{R}^d} \xi(\dot{r}, \ddot{r}) \bar{\eta}_{k,0}(\tilde{y} + \ddot{r} - \ddot{s}) \\
&\quad \left( \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), \lambda_k \ddot{c}^k(\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) \right. \\
&\quad \left. - \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda}{2} \dot{c}(\dot{s} + \dot{r}), \lambda_k \ddot{c}^k(\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) \right) d(\dot{r}, \ddot{r}),
\end{aligned}$$

$$\begin{aligned}
n_k(s, \tilde{y}) &:= \int_{\mathbb{R}^d} \xi(\dot{r}, \ddot{r}) \bar{\eta}_{k,0}(\tilde{y} + \ddot{r} - \ddot{s}) \\
&\quad \left( \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda}{2} \dot{c}(\dot{s} + \dot{r}), \lambda_k \ddot{c}^k(\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) \right. \\
&\quad \left. - \hat{f}_\infty^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda}{2} \dot{c}(\dot{s} + \dot{r}), \lambda_k \ddot{c}^k(\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) \right) d(\dot{r}, \ddot{r})
\end{aligned}$$

and

$$\begin{aligned}
w_k(s) &:= \int_{\mathbb{R}^{d-m}} \int_{\mathbb{R}^d} \xi(\dot{r}, \ddot{r}) \ddot{\eta}_{k,0}(\tilde{y}) \left( \bar{\eta}_{k,0}(\tilde{y}) - \bar{\eta}_{k,0}(\tilde{y} + \ddot{r} - \ddot{s}) \right) \\
&\quad \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2}(\dot{s} + \dot{r}), -\frac{\lambda_k \ddot{c}^k}{2}(\ddot{s} + \ddot{r}), \rho_k, \lambda_k, 0, 0 \right) d(\dot{r}, \ddot{r}) d\tilde{y}.
\end{aligned}$$

Then,

$$\begin{aligned}
(\pi_k(f) - \nu_k(p_{G/U}(f))) \xi(s) &= \int_{\mathbb{R}^{d-m}} q_k(s, \tilde{y}) \ddot{\eta}_{k,0}(\tilde{y}) d\tilde{y} + \int_{\mathbb{R}^{d-m}} u_k(s, \tilde{y}) \ddot{\eta}_{k,0}(\tilde{y}) d\tilde{y} \\
&+ \int_{\mathbb{R}^{d-m}} v_k(s, \tilde{y}) \ddot{\eta}_{k,0}(\tilde{y}) d\tilde{y} + \int_{\mathbb{R}^{d-m}} o_k(s, \tilde{y}) \ddot{\eta}_{k,0}(\tilde{y}) d\tilde{y} \\
&+ \int_{\mathbb{R}^{d-m}} n_k(s, \tilde{y}) \ddot{\eta}_{k,0}(\tilde{y}) d\tilde{y} + w_k(s).
\end{aligned}$$

As in the second case, to show that

$$\| \pi_k(f) - \nu_k(p_{G/U}(f)) \|_{op} \xrightarrow{k \rightarrow \infty} 0,$$

one has to prove that there are  $\kappa_k, \gamma_k, \delta_k, \tau_k, \omega_k$  and  $\epsilon_k$  which are going to 0 for  $k \rightarrow \infty$ , such that

$$\begin{aligned}
\|q_k\|_2 &\leq \kappa_k \|\xi\|_2, \quad \|u_k\|_2 \leq \gamma_k \|\xi\|_2, \quad \|v_k\|_2 \leq \delta_k \|\xi\|_2, \quad \|o_k\|_2 \leq \tau_k \|\xi\|_2, \\
\|n_k\|_2 &\leq \omega_k \|\xi\|_2 \quad \text{and} \quad \|w_k\|_2 \leq \epsilon_k \|\xi\|_2.
\end{aligned}$$

The estimation of the functions  $q_k, u_k, v_k, n_k$  and  $w_k$  is very similar to their estimation in the second case and will thus be skipped. So, it just remains the estimation of  $o_k$ :

For this, first regard the last factor of the function  $o_k$ :

$$\begin{aligned}
& \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2} (\dot{s} + \dot{r}), \lambda_k \ddot{c}^k (\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) \\
& - \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda}{2} \dot{c} (\dot{s} + \dot{r}), \lambda_k \ddot{c}^k (\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) \\
& = \left( \frac{1}{2} (\lambda \dot{c} - \lambda_k \dot{c}^k) (\dot{s} + \dot{r}) \right) \\
& \cdot \int_0^1 \partial_3 \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda}{2} \dot{c} (\dot{s} + \dot{r}) + t \left( \frac{1}{2} (\lambda \dot{c} - \lambda_k \dot{c}^k) (\dot{s} + \dot{r}) \right), \lambda_k \ddot{c}^k (\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) dt.
\end{aligned}$$

Thus, there exists an on  $f$  depending constant  $C_1 > 0$  such that

$$\begin{aligned}
& \left| \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda_k \dot{c}^k}{2} (\dot{s} + \dot{r}), \lambda_k \ddot{c}^k (\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) \right. \\
& \quad \left. - \hat{f}_k^{3,4,5,6,7,8} \left( \dot{s} - \dot{r}, \ddot{s} - \ddot{r}, -\frac{\lambda}{2} \dot{c} (\dot{s} + \dot{r}), \lambda_k \ddot{c}^k (\tilde{y} - \ddot{s}), \rho, \lambda, 0, 0 \right) \right| \\
& \leq \| \lambda \dot{c} - \lambda_k \dot{c}^k \| \| \dot{s} + \dot{r} \| \frac{C_1}{(1 + \| \dot{s} + \dot{r} \|)^{2d+1} (1 + \| \ddot{s} - \ddot{r} \|)^{2d}}.
\end{aligned}$$

Hence, one gets

$$\begin{aligned}
& \| o_k \|_2^2 \\
& = \int_{\mathbb{R}^{d+(d-m)}} |o_k(s, \tilde{y})|^2 d(s, \tilde{y}) \\
& \leq \int_{\mathbb{R}^{d+(d-m)}} \left( \int_{\mathbb{R}^d} |\xi(\dot{r}, \ddot{r})| |\bar{\eta}_{k,0}(\tilde{y} + \ddot{r} - \ddot{s})| \| \lambda \dot{c} - \lambda_k \dot{c}^k \| \| \dot{s} + \dot{r} \| \right. \\
& \quad \left. \frac{C_1}{(1 + \| \dot{s} + \dot{r} \|)^{2d+1} (1 + \| \ddot{s} - \ddot{r} \|)^{2d}} d(\dot{r}, \ddot{r}) \right)^2 d(\dot{s}, \ddot{s}, \tilde{y}) \\
& \stackrel{\text{Cauchy-Schwarz}}{\leq} C_1^2 \| \lambda \dot{c} - \lambda_k \dot{c}^k \|^2 \int_{\mathbb{R}^{d+(d-m)}} \left( \int_{\mathbb{R}^d} \frac{1}{(1 + \| \dot{s} + \dot{r} \|)^{2d} (1 + \| \ddot{s} - \ddot{r} \|)^{2d}} d(\dot{r}, \ddot{r}) \right) \\
& \quad \left( \int_{\mathbb{R}^d} |\xi(\dot{r}, \ddot{r})|^2 |\bar{\eta}_{k,0}(\tilde{y} + \ddot{r} - \ddot{s})|^2 \frac{\| \dot{s} + \dot{r} \|^2}{(1 + \| \dot{s} + \dot{r} \|)^{2d+2} (1 + \| \ddot{s} - \ddot{r} \|)^{2d}} d(\dot{r}, \ddot{r}) \right) d(\dot{s}, \ddot{s}, \tilde{y}) \\
& \leq C_1' \| \lambda \dot{c} - \lambda_k \dot{c}^k \|^2 \int_{\mathbb{R}^{d+(d-m)}} \int_{\mathbb{R}^d} |\xi(\dot{r}, \ddot{r})|^2 |\bar{\eta}_{k,0}(\tilde{y} + \ddot{r} - \ddot{s})|^2 \\
& \quad \frac{1}{(1 + \| \dot{s} + \dot{r} \|)^{2d} (1 + \| \ddot{s} - \ddot{r} \|)^{2d}} d(\dot{r}, \ddot{r}) d(\dot{s}, \ddot{s}, \tilde{y}) \\
& \stackrel{\| \eta_{k,0} \|_2 = 1}{\leq} C_1'' \| \lambda \dot{c} - \lambda_k \dot{c}^k \|^2 \| \xi \|_2^2,
\end{aligned}$$

with matching constants  $C_1' > 0$  and  $C_1'' > 0$ , depending on  $f$ . Hence,  $\tau_k := \sqrt{C_1''} \| \lambda \dot{c} - \lambda_k \dot{c}^k \|$  fulfills  $\tau_k \xrightarrow{k \rightarrow \infty} 0$  and

$$\| o_k \|_2 \leq \tau_k \| \xi \|_2.$$

Thus, for those  $f \in \mathcal{S}(\mathbb{R}^{2d+2+p}) \cong \mathcal{S}(G)$  whose Fourier transform in  $[\mathfrak{g}, \mathfrak{g}]$  has a compact support,

$$\| \pi_k(f) - \nu_k(p_{G/U}(f)) \|_{op} = \sup_{\substack{\xi \in L^2(\mathbb{R}^d) \\ \| \xi \|_2 = 1}} \| (\pi_k(f) - \nu_k(p_{G/U}(f))) (\xi) \|_2 \xrightarrow{k \rightarrow \infty} 0.$$

As in the second case, because of the density in  $C^*(G)$  of the set of Schwartz functions whose partial Fourier transform has a compact support, the claim follows for all  $a \in C^*(G)$ .  $\square$

Now, the assertions for the sequence  $(\pi_k^V)_k$  can be deduced:

#### 4.7.4 Transition to $(\pi_k^V)_k$

Again, because of the equivalence of the representations  $\pi_k$  and  $\pi_k^V$  for every  $k \in \mathbb{N}$ , there exist unitary intertwining operators

$$F_k : \mathcal{H}_{\pi_k^V} \cong L^2(\mathbb{R}^d) \rightarrow \mathcal{H}_{\pi_k} \cong L^2(\mathbb{R}^d) \quad \text{with} \quad F_k \circ \pi_k^V(a) = \pi_k(a) \circ F_k \quad \forall a \in C^*(G).$$

With the injective  $*$ -homomorphism

$$\tau : C_0(\mathbb{R}^{2(d-m)}, \mathcal{K}) \rightarrow C^*(G/U, \chi_\ell), \quad \mathcal{F}(a)|_{L((O_k)_k)} \mapsto p_{G/U}(a)$$

define

$$\tilde{\nu}_k(\varphi) := F_k^* \circ (\nu_k \circ \tau)(\varphi|_{L((O_k)_k)}) \circ F_k \quad \forall \varphi \in CB(S_{i-1}).$$

Then, like in the second case,  $\tilde{\nu}_k$  complies with the demanded requirements and thus, the original representations  $(\pi_k^V)_k$  fulfill Property 3(b).

Finally, one obtains the following result:

#### Theorem 4.5 (Main result).

*The  $C^*$ -algebra  $C^*(G)$  of a connected real two-step nilpotent Lie group is isomorphic (under the Fourier transform) to the set of all operator fields  $\varphi$  defined over  $\widehat{G}$  such that*

1.  $\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$  for every  $i \in \{1, \dots, r\}$  and every  $\gamma \in \Gamma_i$ .
2.  $\varphi \in l^\infty(\widehat{G})$ .
3. The mappings  $\gamma \mapsto \varphi(\gamma)$  are norm continuous on the different sets  $\Gamma_i$ .
4. For any sequence  $(\gamma_k)_{k \in \mathbb{N}} \subset \widehat{G}$  going to infinity  $\lim_{k \rightarrow \infty} \|\varphi(\gamma_k)\|_{\text{op}} = 0$ .
5. For  $i \in \{1, \dots, r\}$  and any properly converging sequence  $\overline{\gamma} = (\gamma_k)_k \subset \Gamma_i$  whose limit set  $L(\overline{\gamma})$  is contained in  $S_{i-1}$  (taking a subsequence if necessary) and for the mappings  $\tilde{\nu}_k = \tilde{\nu}_{\overline{\gamma}, k} : CB(S_{i-1}) \rightarrow \mathcal{B}(\mathcal{H}_i)$  constructed in the preceding sections, one has

$$\lim_{k \rightarrow \infty} \|\varphi(\gamma_k) - \tilde{\nu}_k(\varphi|_{S_{i-1}})\|_{\text{op}} = 0.$$

## 5 Example: The free two-step nilpotent Lie groups of 3 and 4 generators

In the case of the free two-step nilpotent Lie groups of  $n = 3$  and  $n = 4$  generators, the stabilizer of a linear functional  $\ell$ , the in Section 3 constructed polarization  $\mathfrak{p}_\ell^V$ , the coadjoint orbits, as well as the sets  $S_i$  and  $\Gamma_i$  can easily be calculated.

For  $n = 3$ , there are coadjoint orbits of the dimensions 0 and 2 and for  $n = 4$ , the dimensions 0, 2 and 4 appear.

For the free two-step nilpotent Lie groups of 3 generators, the third case regarded in the proof above does not appear: For this, one has to find a sequence of orbits  $(O_k)_k$  whose limit set  $L((O_k)_k)$  consists of orbits of the dimension strictly greater than 0 but strictly smaller than  $\dim(O_k)$ . But as for  $n = 3$  only orbits of the dimensions 0 and 2 appear, such a sequence  $(O_k)_k$  does not exist. However, for the free two-step nilpotent Lie groups of 4 generators, this discussed third case exists.

For both  $n = 3$  and  $n = 4$ , one can also see that the situation occurs where the polarizations  $\mathfrak{p}_\ell^V$  are discontinuous in  $\ell$  on the set  $\{\ell_{O'} \mid O' \in (\mathfrak{g}^*/G)_{2d}\}$ . This shows the necessity of regarding the sets  $\{\ell_{O'} \mid O' \in (\mathfrak{g}^*/G)_{(J,K)}\}$  instead.

Some calculations for the example of the free two-step nilpotent Lie groups of 3 and 4 generators can be found in the doctoral thesis of R.Lahiani (see [4]).

## 6 Appendix

### Lemma 6.1.

*Let  $V$  be a finite-dimensional euclidean vector space and  $S$  an invertible, skew-symmetric endomorphism. Then  $V$  can be decomposed into an orthogonal direct sum of two-dimensional  $S$ -invariant subspaces.*

Proof:

$S$  extends to a complex endomorphism  $S_{\mathbb{C}}$  on the complexification  $V_{\mathbb{C}}$  of  $V$ , which has purely imaginary eigenvalues.

If  $i\lambda \in i\mathbb{R}$  is an eigenvalue, then also  $-i\lambda$  is a spectral element. Denote by  $E_{i\lambda}$  the corresponding eigenspace. These eigenspaces are orthogonal to each other with respect to the Hilbert space structure of  $V_{\mathbb{C}}$  coming from the euclidean scalar product  $\langle \cdot, \cdot \rangle$  on  $V$ .

Let for  $i\lambda$  in the spectrum of  $S_{\mathbb{C}}$

$$V^\lambda := (E_{i\lambda} + E_{-i\lambda}) \cap V.$$

If  $\lambda \neq 0$ ,  $\dim(V^\lambda)$  is even and  $V^\lambda$  is  $S$ -invariant and orthogonal to  $V^{\lambda'}$ , whenever  $|\lambda| \neq |\lambda'|$ :

Indeed, one then has for  $x \in V^\lambda, x' \in V^{\lambda'}$  that

$$\begin{aligned} x + iy &\in E_{i\lambda} \quad \text{and} \quad x - iy \in E_{-i\lambda} \quad \text{for some } y \in V \quad \text{as well as} \\ x' + iy' &\in E_{i\lambda'} \quad \text{and} \quad x' - iy' \in E_{-i\lambda'} \quad \text{for some } y' \in V. \end{aligned}$$

Therefore,

$$\langle x + iy, x' + iy' \rangle = 0 \quad \text{and} \quad \langle x - iy, x' + iy' \rangle = 0.$$

Thus, one has

$$\langle x, x' + iy' \rangle = 0 \quad \text{and hence} \quad \langle x, x' \rangle = 0.$$

Suppose that  $\dim(V^\lambda) > 2$ , choose a vector  $x \in V^\lambda$  of length 1 and let  $y = S(x)$ . Since  $S_{\mathbb{C}}^2 = -\lambda^2 \text{Id}$ , both on  $E_{i\lambda}$  and on  $E_{-i\lambda}$ ,

$$S(y) = S^2(x) = -\lambda^2 x.$$

This shows that  $W_1^\lambda := \text{span}\{x, y\}$  is an  $S$ -invariant subspace of  $V^\lambda$ . If  $V_1^\lambda$  denotes the orthogonal complement of  $W_1^\lambda$  in  $V^\lambda$ , then  $V_1^\lambda$  is  $S$ -invariant, since  $S^t = -S$ .

In this way one can find a decomposition of  $V^\lambda$  into an orthogonal direct sum of two-dimensional  $S$ -invariant subspaces  $W_j^\lambda$  and by summing up over the eigenvalues, one obtains the required decomposition of  $V$ .

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